THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Extra Tutorial 3 (November 16)

Definition. *Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$. For $\delta > 0$, define*

$$\omega_f(\delta) = \sup\{|f(x) - f(y)| : x, y \in A,\ |x - y| < \delta\}.$$

The function so defined $\omega_f : (0, \infty) \to [0, \infty]$ is called the ***modulus of continuity*** *of f over A.*

Remark.

1. *$\omega_f(\cdot)$ is increasing.*
2. *ω_f measures "how uniformly continuous" is the function $f : A \to \mathbb{R}$.*

Exercise 1. *Show that f is uniformly continuous on A if and only if $\lim_{\delta \to 0+} \omega_f(\delta) = 0$.*

Exercise 2. *Prove the following estimates on $\omega_f(\delta)$. Discuss the implication for uniform continuity.*

(a) $f(x) = \sqrt{x}$ and $A = [0, \infty)$. Show that $\omega_f(\delta) \leq \sqrt{\delta}$ for $\delta > 0$.

(b) $f(x) = x^2$ and $A = [0, \infty)$. Show that $\omega_f(\delta) = +\infty$ for $\delta > 0$.

Exercise 3. *Show that the inequality in 2(a) is in fact an equality.*

Exercise 4. *Find the modulus of continuity of the function $\sin(1/x)$ on $(0, 1)$. Hence discuss the uniform continuity of the function.*

Exercise 5 (Extra)**.** *Let $f(x) = x \sin(1/x)$ and $A = (0, \frac{1}{2\pi})$. Show that*

$$\omega_f(\delta) \leq C\delta^{1/2}, \qquad \forall \delta > 0,$$

for some constant $C > 0$ independent of δ.