Selected solution to 2050B Mid-term

3. (i). Note that for $x < 1$ and $M > 0$, one has the following equivalences:

$$\begin{aligned}\frac{x}{x-1} < -M &\Leftrightarrow \frac{x}{1-x} > M \\ &\Leftrightarrow x > M - xM \\ &\Leftrightarrow x(1+M) > M \\ &\Leftrightarrow x > \frac{M}{1+M}.\end{aligned}$$

Suggested by the last equivalence, we would like to have $\frac{M}{1+M} = 1-\delta$, so we set $\delta := 1 - \frac{M}{1+M}$. Note that $\delta > 0$, and if $x \in (1-\delta, 1)$, one has $\frac{M}{1+M} = 1 - \delta < x < 1$, whence $\frac{x}{x-1} < -M$. This shows that

$$\lim_{x\to 1^-} \frac{x}{x-1} = -\infty,$$

because for any $r \in \mathbb{R}$ there exists $M > 0$ such that $-M < r$.

On the other hand, for $x > 1$, we have

$$\frac{x}{x-1} = \frac{x}{|x-1|} > \frac{1}{|x-1|}.$$

Hence, for any $M > 0$, if we set $\delta := \frac{1}{M}$, then for all $x \in (1, 1+\delta)$, we have

$$\frac{x}{x-1} > \frac{1}{\delta} = M.$$

This shows that

$$\lim_{x\to 1^+} \frac{x}{x-1} = \infty.$$

Finally, since

$$\lim_{x\to 1^-} \frac{x}{x-1} \neq \lim_{x\to 1^+} \frac{x}{x-1},$$

we see that $\lim_{x\to 1} \frac{x}{x-1}$ does not exist.*

*An alternative approach for question 3(i) is to use the result of question 2(ii).

(ii). (It seems that the function in consideration is continuous, so we guess that the limit is $\sqrt{x_0^2+1}$)

Firstly, note that

$$\begin{aligned}\left|\sqrt{x^2+1}-\sqrt{x_0^2+1}\right| &= \left|\frac{x^2-x_0^2}{\sqrt{x^2+1}+\sqrt{x_0^2+1}}\right| \\ &= \frac{|x-x_0|\,|x+x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}}.\end{aligned}$$

By the elementary inequality $\frac{|x|}{\sqrt{x^2+1}} \leq 1$, we have

$$\begin{aligned}\frac{|x+x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} &\leq \frac{|x|+|x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} \\ &= \frac{|x|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} + \frac{|x_0|}{\sqrt{x^2+1}+\sqrt{x_0^2+1}} \\ &\leq \frac{|x|}{\sqrt{x^2+1}} + \frac{|x_0|}{\sqrt{x_0^2+1}} \\ &\leq 1+1=2.\end{aligned}$$

Therefore

$$\left|\sqrt{x^2+1}-\sqrt{x_0^2+1}\right| \leq 2\,|x-x_0|\,,$$

which is nice enough for us to apply the ε-δ terminology.

Let $\varepsilon > 0$. For this ε, we set $\delta_\varepsilon := \varepsilon/2$. Now whenever x satisfies $0 < |x-x_0| < \delta_\varepsilon$, we have

$$\begin{aligned}\left|\sqrt{x^2+1}-\sqrt{x_0^2+1}\right| &\leq 2\,|x-x_0| \\ &< 2\cdot\delta_\varepsilon = \varepsilon.\end{aligned}$$

By ε-δ terminology, we conclude that

$$\lim_{x\to x_0}\sqrt{x^2+1} = \sqrt{x_0^2+1}.$$

(iii). Let $\varepsilon > 0$. Set $\delta := \min(1, \frac{\varepsilon}{2(5+M)})$, where

$$M := (|x_0| + 1)^2 + (|x_0| + 1)\,|x_0| + |x_0|^2\,.$$

Let $0 < |x - x_0| < \delta$. One checks $|f(x) - f(x_0)| < \varepsilon$ where $f(x) = x^3 - 5x - 7$:

$$\begin{aligned}
&\left|(x^3 - 5x - 7) - (x_0^3 - 5x_0 - 7)\right| \\
&\leq \left|x^3 - x_0^3\right| + 5\,|x - x_0| \\
&= |x - x_0|\,\left|x^2 + xx_0 + x_0^2\right| + 5\,|x - x_0| \\
&\leq (M + 5)\,|x - x_0| \quad (\text{as } |x - x_0| < \delta \leq 1 \text{ so } |x| < |x_0| + 1\ ) \\
&\leq \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

By ε-δ terminology, we conclude that

$$\lim_{x \to x_0} (x^3 - 5x - 7) = x_0^3 - 5x_0 - 7.$$

Second approach:

Firstly, we have the following result:

$$\lim_{x \to x_0} (f_1(x) f_2(x)) = (\lim_{x \to x_0} f_1(x)) \cdot (\lim_{x \to x_0} f_2(x))$$

if $\lim_{x \to x_0} f_i(x)$ exists.

Therefore, since $\lim_{x \to x_0} x$ exists and equals x_0, we have

$$\lim_{x \to x_0} x^2 = x_0^2,$$

and so

$$\lim_{x \to x_0} x^3 = (\lim_{x \to x_0} x^2) \cdot (\lim_{x \to x_0} x) = x_0^3.$$

Similarly, since $\lim_{x \to x_0}(-5) = -5$, the foregoing result for limits gives

$$\lim_{x \to x_0} -5x = -5x_0.$$

Next, we have the following result:

$$\lim_{x\to x_0}(f_1(x)+f_2(x)) = \lim_{x\to x_0} f_1(x) + \lim_{x\to x_0} f_2(x)$$

if $\lim_{x\to x_0} f_i(x)$ exists.

Therefore, since $\lim_{x\to x_0} x^3$ and $\lim_{x\to x_0} -5x$ exists, we have

$$\lim_{x\to x_0}(x^3-5x) = \lim_{x\to x_0} x^3 + \lim_{x\to x_0}(-5x) = x_0^3 - 5x_0.$$

Finally, since $\lim_{x\to x_0}(-7) = -7$, by applying the foregoing result once more, we have

$$\lim_{x\to x_0}(x^3-5x-7) = \lim_{x\to x_0}(x^3-5x) + \lim_{x\to x_0}(-7) = x_0^3 - 5x_0 - 7.$$

5. We have $\liminf_n x_n = \lim_{n\to\infty} y_n$, where y_n is defined by $y_n := \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$.

Brief explanation (FYR only, need not be given in the answer):

Since

$$\begin{aligned}\{x_1,x_2,x_3,x_4,\ldots\} &\supseteq \{x_2,x_3,x_4,\ldots\}\\ &\supseteq \{x_3,x_4,\ldots\}\\ &\supseteq \cdots,\end{aligned}$$

therefore

$$\begin{aligned}\inf\{x_1,x_2,x_3,x_4,\ldots\} &\le \inf\{x_2,x_3,x_4,\ldots\}\\ &\le \inf\{x_3,x_4,\ldots\}\\ &\le \cdots.\end{aligned}$$

Define $y_n := \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. By above we see that (y_n) is an increasing sequence. It is bounded above as well, because (x_n) is a bounded sequence. By monotone convergence theorem, $\lim_{n\to\infty} y_n$ exists. We take

$$\liminf_n x_n = \lim_{n\to\infty} y_n.$$