TA's solution to 2050B assignment 1

1. (a). (2 marks)

We have

$$\begin{aligned}
0 &= a \cdot 0 + (-(a \cdot 0)) \quad \text{(existence of an additive inverse of } (a \cdot 0))\\
&= a \cdot (0+0) + (-(a \cdot 0)) \quad \text{(property of a zero element)}\\
&= (a \cdot 0 + a \cdot 0) + (-(a \cdot 0)) \quad \text{(distributive property of } \times \text{ over } +)\\
&= a \cdot 0 + (a \cdot 0 + (-(a \cdot 0)) \quad \text{(associative property of } +)\\
&= a \cdot 0 + 0 \quad \text{(property of } -(a \cdot 0))\\
&= a \cdot 0. \quad \text{(property of a zero element)}
\end{aligned}$$

(b). (2 marks)

By quiz 1, we know that every element r in $\mathbb{R}$ has a unique additive inverse (denoted by $-r$). Assuming this, the question asks us to show that:

$$\text{the unique additive inverse of } a = \text{(the unique additive inverse of 1)} \times a.$$

If we can show that $a + (-1) \cdot a = 0$, then by commutativity of addition, we have $(-1) \cdot a + a = 0$ as well, so by the uniqueness of additive inverse of a, we have $(-1) \cdot a = -a$. Now

$$\begin{aligned}
a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \quad \text{(property of a unit element)}\\
&= (1 + (-1)) \cdot a \quad \text{(distributive property of } \times \text{ over } +)\\
&= 0 \cdot a \quad \text{(property of the unique additive inverse of 1)}\\
&= a \cdot 0 \quad \text{(commutative property of of } \times)\\
&= 0. \quad \text{(by the result of 1(a))}
\end{aligned}$$

The result follows.

(c). The question asks us to show that:

$$a = \text{the additive inverse of the additive inverse of } a.$$

Using the result of 1(b),

$$\begin{aligned}
&\text{the additive inverse of the additive inverse of } a \\
&= (-1) \times (\text{the additive inverse of } a) \\
&= (-1) \times [(-1) \times a)] \\
&= [(-1) \times (-1)] \times a.
\end{aligned}$$

Therefore, the proof is finished if we can show

$$1 = (-1) \times (-1).$$

Note that

$$\begin{aligned}
(-1) \cdot (-1) &= 0 + (-1) \cdot (-1) \\
&= [1 + (-1)] + (-1) \cdot (-1) \\
&= 1 + [(-1) + (-1) \cdot (-1)] \\
&= 1 + [(-1) \cdot 1 + (-1) \cdot (-1)] \\
&= 1 + [(-1) \cdot (1 + (-1))] \\
&= 1 + ((-1) \cdot 0) \\
&= 1 + 0 \quad (\text{by the result of } 1(\text{a})) \\
&= 1.
\end{aligned}$$

Done.

(d). The question asks us to show that:

$$(\text{the additive inverse of } a) \times (\text{the additive inverse of } b) = a \times b.$$

This follows from

$$\begin{aligned}
(-a) \cdot (-b) &= [(-1) \cdot a] \cdot [(-1) \cdot b] \quad (\text{by the result of } 1(\text{b})) \\
&= \{[(-1) \cdot a] \cdot (-1)\} \cdot b \\
&= \{(-1) \cdot [a \cdot (-1)]\} \cdot b \\
&= \{(-1) \cdot [(-1) \cdot a]\} \cdot b \\
&= \{[(-1) \cdot (-1)] \cdot a\} \cdot b \\
&= \{1 \cdot a\} \cdot b \quad (\text{by the proof of } 1(\text{c})) \\
&= a \cdot b.
\end{aligned}$$

(e). If $a = 0$, then by the result of 1(a) we have $a^2 = a \cdot 0 = 0$. Else if $a \neq 0$, please refer to textbook 2.1.8 Theorem.

(f). Please refer to textbook 2.1.7 Theorem.

(g). Please refer to textbook 2.1.13 Examples.

2. (a). (2 marks)

($\Rightarrow$) Suppose $|x - a| < \epsilon$. This implies

$$\begin{cases} |x-a| = x-a \\ x-a < \epsilon \end{cases} \text{or} \begin{cases} |x-a| = -(x-a) \\ -(x-a) < \epsilon, \end{cases}$$

so

$$\begin{cases} 0 \leq x-a \\ x-a < \epsilon \end{cases} \text{or} \begin{cases} x-a \leq 0 \\ -(x-a) < \epsilon, \end{cases}$$

so

$$0 \leq x - a < \epsilon \text{ or } -\epsilon < x - a \leq 0.$$

In the first case, we have $-\epsilon < 0 \leq x - a < \epsilon$. In the second case, we have $-\epsilon < x-a \leq 0 < \epsilon$. Therefore we see that $-\epsilon < x-a < \epsilon$ is always true under the assumption $|x - a| < \epsilon$.

($\Leftarrow$) Suppose $-\epsilon < x - a < \epsilon$.
If $0 \leq x - a$, then $|x - a| = x - a < \epsilon$.
Else if $x - a < 0$, then $-\epsilon < x - a < 0$, so $0 < -(x - a) < \epsilon$, therefore $|x - a| = -(x - a) < \epsilon$.
Hence, in all cases we have $|x - a| < \epsilon$.

(b).

$$|x - 1| > |x + 1|$$

$\Leftrightarrow$

$$\begin{cases} x-1 \geq 0 \\ x+1 \geq 0 \\ x-1 > x+1 \end{cases} \text{or} \begin{cases} -(x-1) \geq 0 \\ x+1 \geq 0 \\ -(x-1) > x+1 \end{cases}$$
$$\text{or} \begin{cases} x-1 \geq 0 \\ -(x+1) \geq 0 \\ x-1 > -(x+1) \end{cases} \text{or} \begin{cases} -(x-1) \geq 0 \\ -(x+1) \geq 0 \\ -(x-1) > -(x+1) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x \geq 1 \\ -1 > 1 \end{cases} \text{or} \begin{cases} 1 \geq x \geq -1 \\ 0 > 2x \end{cases} \text{or} \begin{cases} -1 \geq x \geq 1 \\ 2x > 0 \end{cases} \text{or} \begin{cases} -1 \geq x \\ 1 > -1 \end{cases}$$

$\Leftrightarrow$

$$0 > x \geq -1 \text{ or } -1 \geq x.$$

We conclude that $0 > x$ are exactly those real numbers which satisfy $|x-1| > |x+1|$.

3 (a). (i). (0.8 marks)
ℓ is a lower bound of A iff no element in A is less than ℓ.

(ii). (1.2 marks)
ℓ is not a lower bound of A iff there is an element a_0 in A such that a_0 is less than ℓ.

(b). (i). A is bounded below iff there exists a real number L_0 such that $L_0 \leq a$ for all $a \in A$.

(ii). A is not bounded below iff for whatever real number λ (no matter how negative λ is), we can still find an element α in A such that $\alpha < \lambda$.

4. (a). Please refer to textbook 3.2.2 Theorem.

(b). (2 marks) Please refer to textbook 3.2.3 Theorem, which is very well-written.

(c). Please refer to textbook 3.2.3 Theorem.