MATH2050B Mathematical Analysis I

Homework 3 suggested Solution*

Question 4. Let

$$x_{n+1} = 2 + \frac{x_n}{2}, \qquad \forall n \in \mathbb{N}.$$

Then, for each of the following cases, show that (x_n) converges (any find the value of the limit:

(i) $x_1 = 0$;

(ii) $x_1 = 10$. (Hint: Can the MCT be applied?)

Solution:

Method 1:

(i) Let $P(n)$ denote the proposition that $x_{n+1} \geq x_n$ and $x_n \leq 4$.

Notice that when $n = 1$, we have $0 = x_1 < 4$ and $2 = x_2 > x_1$, thus $P(1)$ is true.

Suppose $P(n)$ is true, i.e. $x_{n+1} \geq x_n$ and $x_n \leq 4$. It follows that

$$x_{n+1} = 2 + \frac{x_n}{2} \leq 2 + \frac{4}{2} = 4;$$

$$x_{n+2} = 2 + \frac{x_{n+1}}{2} \geq \frac{x_{n+1}}{2} + \frac{x_{n+1}}{2} = x_{n+1};$$

Hence $P(n+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is monotone increasing and bounded above. By MCT, we have $\{x_n\}$ is convergent.

By the contruction of sequence $\{x_n\}$, we thus get

$$\lim_{n\to\infty} x_{n+1} = 2 + \frac{\lim_{n\to\infty} x_n}{2},$$

which implies $\lim_{n\to\infty} x_n = 4$.

(ii) It is obvious that $x_n \geq 0$ for all $n \in \mathbb{N}$. The argument is similar to (i). Let $S(n)$ denote the proposition that $x_{n+1} \leq x_n$. Notice that when $n = 1$, we have $x_2 = 2 + \frac{x_1}{2} \leq x_1$, thus $S(1)$ is true.

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Suppose $S(n)$ is true, i.e. $x_{n+1} \le x_n$. It follows that

$$x_{n+2} = 2 + \frac{x_{n+1}}{2} \le \frac{x_{n+1}}{2} + \frac{x_{n+1}}{2} = x_{n+1};$$

Hence $S(n+1)$ is true. By the principle of mathematical induction, $S(n)$ is true for all $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is monotone decreasing and bounded below by 0. By MCT, we have $\{x_n\}$ is convergent.

By the contruction of sequence $\{x_n\}$, we thus get

$$\lim_{n\to\infty} x_{n+1} = 2 + \frac{\lim_{n\to\infty} x_n}{2},$$

which implies $\lim_{n\to\infty} x_n = 4$.

Method 2: Since $x_{n+1} = 2 + \frac{x_n}{2}$, that is, $x_{n+1} - 4 = \frac{1}{2}(x_n - 4)$. Define a sequence $y_n := x_n - 4$, for all $n \in \mathbb{N}$. This yields that

$$y_n = 2^{-n+1} y_1.$$

Therefore, we have $\lim_{n\to\infty} y_n = 0$, that is, $\lim_{n\to\infty} (x_n - 4) = 0$, hence that $\lim_{n\to\infty} x_n = 4$.

Question 5. Show that $\lim_{n\to\infty} \frac{n^7}{(1+\delta)^n} = 0$ (where $\delta > 0$) .

Hint (similar to Q1 but expand more terms when apply the Binomial).

Solution:

For $n \in \mathbb{N}$, we note that

$$\begin{aligned} \frac{n^7}{(1+\delta)^n} &= \frac{n^7}{1 + C_1^n \delta + C_2^n \delta^2 + \cdots + C_k^n \delta^k + \cdots + \delta^n} \\ &\le \frac{n^7}{C_8^n \delta^8} \\ &= \frac{8! \cdot n^7}{n(n-1)(n-2)\cdots(n-7)\delta^8} \\ &= \frac{8!}{n(1-1/n)(1-2/n)\cdots(1-7/n)\delta^8} \end{aligned} \tag{1}$$

Notice that $\lim_{n\to\infty} \frac{8!}{n(1-1/n)(1-2/n)\cdots(1-7/n)\delta^8} = 0$, by computation rules, we thus get $\lim_{n\to\infty} \frac{n^7}{(1+\delta)^n} = 0$.

Question 6. Let $x_1 > 0$ and

$$x_{n+1} = x_n + \frac{1}{x_1} \qquad \forall n \in \mathbb{N}.$$

Use two methods below to show that (x_n) does not converge:

(a) Use $Q6$ of HW 2.

(b) Use (algetrvaic computation rules).

Solution:

(a) By the contruction of the sequence and $x_1 > 0$, we have $\{x_n\}$ is a monotone increasing sequence. It follows that

$$x_{n+1} = x_n + \frac{1}{x_1} \geq x_n + \frac{1}{x_n}.$$

By *Q*6 of HW2 we see that any sequence $\{y_n\}$ with $y_{n+1} = y_n + \frac{1}{y_n}$ is not bouded above. Therefore, the seuqence $\{x_n\}$ is also not bounded above, hence that $\{x_n\}$ is not convergent.

(b) Suppose on the contrary that $\lim_{n\to\infty} x_n = \ell$, for some $\ell \in \mathbb{R}$. Note that $x_{n+1} = x_n + \frac{1}{x_1}$, by taking limits on both sides, we have

$$\ell = \ell + \frac{1}{x_1},$$

which contracts with $x_1 > 0$. Therefore $\{x_n\}$ does not converge.

Question 7. Suppose $\lim_n y_n = y$. Show

(i) If $y > 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot y < y_n < 2y, \qquad \forall n \geqslant N.$$

(ii) If $y \neq 0$ then there exists $N \in \mathbb{N}$ such that

$$0.9 \cdot |y| < |y_n| < 2|y|, \qquad \forall n \geqslant N.$$

(iii) Suppose $\lim_n y_n = y, y \neq 0$ and $\delta \in (0, |y|)$. Then $\exists N \in \mathbb{N}$ s.t.

$$(1-\delta)|y| < |y_n| < \frac{1}{3} + |y| \qquad \forall n \geqslant N.$$

Solution:

(i) Since $y \neq 0$, we have $\lim_n y_n = y$ if and only if $\lim_n \frac{y_n}{y} = 1$. This yields that there exists $N \in \mathbb{N}$ such that

$$|\frac{y_n}{y} - 1| < \frac{1}{10}, \quad \text{for all } n \geqslant N.$$

That is

$$\frac{9}{10} < \frac{y_n}{y} < \frac{11}{10} < 2, \quad \text{for all } n \geqslant N,$$

which is equivalent to $\frac{9}{10}y < y_n < 2y$ ($\forall\ n \geqslant N$), due to the fact that $y > 0$.

(ii) It follows from (i) that for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|\frac{y_n}{y} - 1| < \epsilon, \quad \text{for all } n \geqslant N_1.$$

Notice that $|\frac{|y_n|}{|y|}-1| \le |\frac{y_n}{y}-1|$, we thus get $\lim_n \frac{|y_n|}{|y|}=1$. We now apply argument in (i) again, with $\{y_n\}$ replaced by $\{|y_n|\}$, to obtain that there exists $N_2 \in \mathbb{N}$ such that

$$0.9 \cdot |y| < |y_n| < 2|y|, \qquad \forall n \geqslant N_2.$$

(iii) It follows from the proof of (ii) that there exists $N_3 \in \mathbb{N}$ such that

$$\left|\frac{|y_n|}{|y|}-1\right| < \delta, \qquad \text{for all } n \ge N_3. \tag{2}$$

This implies $(1-\delta)|y| < |y_n|$ for all $n \ge N_3$.

On the other hand, we can see that $\lim_n y_n = y$ implies $\lim_n |y_n| = |y|$, due to the fact that $||y_n|-|y|| \le |y_n - y|$ for any $n \in \mathbb{N}$. It follows that that there exists $N_4 \in \mathbb{N}$ such that

$$||y_n|-|y|| < \frac{1}{3}, \qquad \text{for all } n \ge N_4. \tag{3}$$

This implies $|y_n| < |y| + \frac{1}{3}$ for all $n \ge N_4$. Let $N' = \max\{N_3, N_4\}$. Combining inequalies (2) and (3), we get

$$(1-\delta)|y| < |y_n| < \frac{1}{3} + |y|, \qquad \text{for all } n \ge N'.$$