# Solution 1

1.* (3rd: P.12, Q6)

Let $f : X \to Y$ be a mapping of a nonempty space $X$ into $Y$. Show that $f$ is one-to-one if and only if there is a mapping $g : Y \to X$ such that $g \circ f$ is the identity map on $X$, that is, such that $g(f(x)) = x$ for all $x \in X$.

**Solution.** Suppose $f$ is one-to-one. Fix $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x & \text{if } y \in f[X] \text{ and } f(x) = y, \\ x_0 & \text{otherwise.} \end{cases}$$

$g$ is a well-defined mapping since $f$ is one-to-one. It is easy to see that $g \circ f$ is the identity map on $X$.

On the other hand, suppose that such mapping $g$ exists. If $f(x_1) = f(x_2)$, $x_1, x_2 \in X$, then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

Hence $f$ is one-to-one. ◀

2. (3rd: P.12, Q7)

Let $f : X \to Y$ be a mapping of $X$ into $Y$. Show that $f$ is onto if there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map in $Y$, that is, $f(g(y)) = y$ for all $y \in Y$.

**Solution.** Suppose that such mapping $g$ exists. For any $y \in Y$, $x := g(y) \in X$ satisfies

$$f(x) = f(g(y)) = y.$$

Hence $f$ is onto. ◀

3. Show that any set $X$ can be "indexed": $\exists$ a set $I$ and a function $f : I \to X$ such that $\{f(i) : i \in I\} = X$.

**Solution.** Simply take $I = X$ and $f : I \to X$ to be the identity function. ◀

4.* (3rd: P.16, Q14)

Given a set $B$ and a collection of sets $\mathcal{C}$. Show that

$$B \cap \left[ \bigcup_{A \in \mathcal{C}} A \right] = \bigcup_{A \in \mathcal{C}} (B \cap A).$$

**Solution.**

$$\begin{aligned} x \in B \cap \left[ \bigcup_{A \in \mathcal{C}} A \right] &\iff x \in B \text{ and } x \in \bigcup_{A \in \mathcal{C}} A \\ &\iff x \in B \text{ and } (\exists\, A)(A \in \mathcal{C} \text{ and } x \in A) \\ &\iff (\exists\, A)(x \in B \text{ and } (A \in \mathcal{C} \text{ and } x \in A)) \\ &\iff (\exists\, A)(A \in \mathcal{C} \text{ and } x \in A \cap B) \\ &\iff x \in \bigcup_{A \in \mathcal{C}} (B \cap A). \end{aligned}$$

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5. (3rd: P.16, Q15)

Show that if $\mathcal{A}$ and $\mathcal{B}$ are two collections of sets, then

$$[\bigcup\{A : A \in \mathcal{A}\}] \cap [\bigcup\{B : B \in \mathcal{B}\}] = \bigcup\{A \cap B : (A, B) \in \mathcal{A} \times \mathcal{B}\}.$$

**Solution.** Using the result in Q4 twice, we have

$$[\bigcup\{A : A \in \mathcal{A}\}] \cap [\bigcup\{B : B \in \mathcal{B}\}] = \bigcup_{B \in \mathcal{B}} [\bigcup\{A : A \in \mathcal{A}\}] \cap B$$

$$= \bigcup_{B \in \mathcal{B}} \left[ \bigcup_{A \in \mathcal{A}} (A \cap B) \right] = \bigcup_{(A,B) \in \mathcal{A} \times \mathcal{B}} (A \cap B) = \bigcup\{A \cap B : (A, B) \in \mathcal{A} \times \mathcal{B}\}.$$

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6. (3rd: P.16, Q16)

Let $f : X \to Y$ be a function and $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of $X$.

(a) Show that $f[\bigcup A_\lambda] = \bigcup f[A_\lambda]$.

(b) Show that $f[\bigcap A_\lambda] \subset \bigcap f[A_\lambda]$.

(c) Give an example where $f[\bigcap A_\lambda] \neq \bigcap f[A_\lambda]$.

**Solution.** (a)

$$\begin{aligned} y \in f\left[\bigcup A_\lambda\right] &\iff (\exists\, x) \left(y = f(x) \text{ and } x \in \bigcup A_\lambda\right) \\ &\iff (\exists\, x)\, [y = f(x) \text{ and } (\exists\, \lambda)(x \in A_\lambda)] \\ &\iff (\exists\, x)\, (\exists\, \lambda)\, (y = f(x) \text{ and } x \in A_\lambda) \\ &\iff (\exists\, \lambda)\, (\exists\, x)\, (y = f(x) \text{ and } x \in A_\lambda) \\ &\iff (\exists\, \lambda)\, (y \in f[A_\lambda]) \\ &\iff y \in \bigcup f[A_\lambda]. \end{aligned}$$

(b)

$$\begin{aligned} y \in f\left[\bigcap A_\lambda\right] &\iff (\exists\, x) \left(y = f(x) \text{ and } x \in \bigcap A_\lambda\right) \\ &\iff (\exists\, x)\, [y = f(x) \text{ and } (\forall\, \lambda)(x \in A_\lambda)] \\ &\iff (\exists\, x)\, (\forall\, \lambda)\, (y = f(x) \text{ and } x \in A_\lambda) \\ &\implies (\forall\, \lambda)\, (\exists\, x)\, (y = f(x) \text{ and } x \in A_\lambda) \\ &\iff (\forall\, \lambda)\, (y \in f[A_\lambda]) \\ &\iff y \in \bigcap f[A_\lambda]. \end{aligned}$$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $A = (-\infty, 0)$ and $B = (0, \infty)$. Then $f(A \cap B) = f(\emptyset) = \emptyset$ while $f(A) \cap f(B) = (0, \infty) \cap (0, \infty) = (0, \infty)$.

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7.* (3rd: P.16, Q17)

Let $f : X \to Y$ be a function and $\{B_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of $Y$.

(a) Show that $f^{-1}[\bigcup B_\lambda] = \bigcup f^{-1}[B_\lambda]$.
(b) Show that $f^{-1}[\bigcap B_\lambda] = \bigcap f^{-1}[B_\lambda]$.
(c) Show that $f^{-1}[B^c] = (f^{-1}[B])^c$ for $B \subset Y$.

**Solution.** (a)

$$\begin{aligned} x \in f^{-1}\left[\bigcup B_\lambda\right] &\iff f(x) \in \bigcup B_\lambda \\ &\iff (\exists\, \lambda)(f(x) \in B_\lambda) \\ &\iff (\exists\, \lambda)(x \in f^{-1}[B_\lambda]) \\ &\iff x \in \bigcup f^{-1}[B_\lambda]. \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}\left[\bigcap B_\lambda\right] &\iff f(x) \in \bigcap B_\lambda \\ &\iff (\forall\, \lambda)(f(x) \in B_\lambda) \\ &\iff (\forall\, \lambda)(x \in f^{-1}[B_\lambda]) \\ &\iff x \in \bigcap f^{-1}[B_\lambda]. \end{aligned}$$

(c)

$$\begin{aligned} x \in f^{-1}[B^c] &\iff f(x) \in B^c \\ &\iff \neg(f(x) \in B) \\ &\iff \neg(x \in f^{-1}[B]) \\ &\iff x \in (f^{-1}[B])^c. \end{aligned}$$

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8.* (3rd: P.16, Q18)

(a) Show that if $f$ maps $X$ into $Y$ and $A \subset X$, $B \subset Y$, then

$$f[f^{-1}[B]] \subset B$$

and

$$f^{-1}[f[A]] \supset A.$$

(b) Give examples to show that we need not have equality.
(c) Show that if $f$ maps $X$ onto $Y$ and $B \subset Y$, then

$$f[f^{-1}[B]] = B.$$

**Solution.** (a) It is easy to see that

$$\begin{aligned} y \in f[f^{-1}[B]] &\iff (\exists\, x)(y = f(x) \text{ and } x \in f^{-1}[B]) \\ &\iff (\exists\, x)(y = f(x) \text{ and } f(x) \in B) \\ &\implies y \in B, \end{aligned}$$

and

$$\begin{aligned} x \in A &\implies f(x) \in f[A] \\ &\iff x \in f^{-1}[f[A]]. \end{aligned}$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $A = [0, \infty)$ and $B = (-\infty, \infty)$. Then

$$f[f^{-1}[B]] = f[(-\infty, \infty)] = [0, \infty) \subsetneq B$$

while

$$f^{-1}[f[A]] = f^{-1}[[0, \infty)] = (-\infty, \infty) \supsetneq A.$$

(c) Suppose $f$ maps $X$ onto $Y$. Let $y \in B$. Since $f$ is onto, there exists $x \in X$ such that $f(x) = y$. As $y \in B$, we have $x \in f^{-1}[B]$. Hence $y = f(x) \in f[f^{-1}[B]]$. Therefore $f[f^{-1}[B]] \supset B$.

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9. Show that $f \mapsto \int_0^1 f(x)dx$ is a "monotone" function on $\mathcal{R}[0, 1]$ (consisting of all Riemann integrable functions on $[0, 1]$), and $\mathcal{R}[0, 1]$ is a linear space. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

if $f, f_n \in \mathcal{R}[0, 1]$ such that

$$\lim_{n\to\infty} ( \sup_{x\in[0,1]} |f_n(x) - f(x)|) = 0.$$

**Solution.** It is easy to verify that $\mathcal{R}[0, 1]$ is a linear space and that $\phi : f \mapsto \int_0^1 f(x)dx$ is a linear, monotone function on $\mathcal{R}[0, 1]$. We only prove the limit equality.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon.$$

The monotonicity and linearity of the function $\phi : f \mapsto \int_0^1 f(x)dx$ imply that

$$\left| \int_0^1 f_n(x)dx - \int_0^1 f(x)dx \right| = \left| \int_0^1 (f_n(x) - f(x))\, dx \right| \leq \int_0^1 |f_n(x) - f(x)|\, dx.$$

Again by the monotonicity of $\phi$, we have, for all $n \geq N$,

$$\begin{aligned} \left| \int_0^1 f_n(x)dx - \int_0^1 f(x)dx \right| &\leq \int_0^1 \sup_{x\in[0,1]} |f_n(x) - f(x)|\, dx \\ &\leq \int_0^1 \varepsilon\, dx = \varepsilon. \end{aligned}$$

This completes the proof of the limit equality.

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