4050 Hw 3

1. Show that $\inf X \geq \inf Y$ whenever $X \subseteq Y$ ($\subseteq \mathbb{R}$) and hence that $m^*(A) \uparrow$ (i.e. $m^*(A) \leq m^*(B)$ if $A \subseteq B$ ($\subseteq \mathbb{R}$)).

2. Let $\mathcal{A}$ be an algebra of subsets of X. Show that $\mathcal{A}$ is an σ-algebra if (and only if) $\mathcal{A}$ is stable w.r.t countable disjoint unions:
$\bigcup_{n=1}^{\infty} A_n$ whenever $A_n \in \mathcal{A}$ $\forall n \in \mathbb{N}$ and $A_m \cap A_n = \emptyset$ $\forall m \neq n$.

3. Suppose $[a,b]$ ($\subseteq \mathbb{R}$) is covered by a finite family $\mathcal{C}$ of open intervals. Show that $b-a \leq$ sum of lengths of intervals in $\mathcal{C}$ (by MI to $n := \#(\mathcal{C})$, the number of elements of $\mathcal{C}$).

4. (cf. Royden 3rd, P52, Q51). Upper/Lower Envelopes of $f: [a,b] \to \mathbb{R}$. Define $h, g: [a,b] \to [-\infty, \infty]$ by, $\forall y \in [a,b]$,
$h(y) := \inf\{h_\delta(y) : \delta > 0\}$, where $h_\delta(y) := \sup\{f(x) : x \in [a,b], |x-y| < \delta\}$
$g(y) := \sup\{g_\delta(y) : \delta > 0\}$, where $g_\delta(y) := \inf\{f(x) : x \in [a,b], |x-y| < \delta\}$.
Prove the following:
   a. $g \leq f \leq h$ pointwisely on $[a,b]$, and $\forall x \in [a,b]$,
   $g(x) = f(x)$ iff f is lsc at x ($f(x) = h(x)$ iff f is usc at x)
   so $g(x) = h(x)$ iff f is continuous at x.
   b. If f is bounded (so g, h are real-valued) then g is lsc & h is usc
   c. If φ is a lsc on $[a,b]$ such that $\varphi \leq f$ (pointwise) on $[a,b]$ then $\varphi \leq g$. State and show the corresponding result for h.
   d. Let $C_n := \{x \in [a,b] : h(x) - g(x) < \frac{1}{n}\}$ $\forall n \in \mathbb{N}$. Then $C := \bigcap_{n=1}^{\infty} C_n$ is exactly the set of all continuity points of f and is a G_δ-set.

Note. More suggestive notations for g, h are $\underline{f}$ and $\overline{f}$.

5. Let $f: [a,b] \to [m,M]$. For each $P \in \text{par}[a,b]$, let $u(f;P)$ and $U(f;P)$ denote the lower/upper Riemann-sum-functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}$ $\forall n$ and $\|P_n\| \to$ (the max of subinterval-lengths of P_n). Show that, $\forall x \in [a,b] \setminus A$
$\lim_n (u(f;P_n))(x) = \underline{f}(x)$ and $\lim_n (U(f;P_n))(x) = \overline{f}(x)$, where A denotes the union of all end-points of P_n $\forall n \in \mathbb{N}$