THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 11 (November 25, 27)

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \qquad \text{for all } x, u \in A, \tag{$*$}$$

then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Remarks. When A is an interval I, the condition $(*)$ means that the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem. *If $f\colon A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Example 1. (a) $f(x) := x^2$ is a Lipschitz function on $[0, b]$, $b > 0$, but does not satisfy a Lipschitz condition on $[0, \infty)$.

(b) $g(x) := \sqrt{x}$ is uniformly continuous on $[a, \infty)$, $a > 0$, but not a Lipschitz function on $[0, \infty)$.

Example 2. Let A be a nonempty subset of $\mathbb{R}$. For $x \in \mathbb{R}$, define

$$\rho_A(x) = \inf\{|x - y| : y \in A\}.$$

(a) Show that ρ_A is Lipschitz on A, hence uniformly continuous on A.

(b) Show that $\rho_A(x) = 0$ if and only if $x \in \overline{A}$.

Example 3. Let f and g be uniformly continuous on $A \subseteq \mathbb{R}$.

(a) Is fg uniformly continuous on A?

(b) If f, g are both bounded on A, show that fg is uniformly continuous on A.

(c) If $A = (0, 1)$, show that fg uniformly continuous on A.