MATH2050B 2021 HW 6

TA's solutions[1] to selected problems

Q1. Show that for $\varnothing \neq A \subset \mathbb{R}$ and $c \in \mathbb{R}$ that

(i) $c \in A^c$ if and only if each neighborhood of c intersect $A \setminus \{c\}$.

(ii) Let $c \in [-\infty, \infty]$, and let $U_1, \dots U_n$ be neighborhoods of c. Then $\cap_{i=1}^n U_i$ is a neighborhood of c.

(iii) (ii) is not true for infinitely many neighborhoods.

Solution. (i): is due to definition, see Q3 of HW5.

(ii): **Case 1.** $c \in \mathbb{R}$. Then for each $i = 1, 2, \dots, n$ there exists $r_i > 0$ such that $V_{r_i}(c) \subset U_i$. Take $r = \min(r_1, r_2, \dots, r_n) > 0$. Then $V_r(c) \subset U_i$ for all i.

Case 2. $c = \infty$. Then for each i there exists $r_i \in \mathbb{R}$ such that $V_{r_i}(\infty) \subset U_i$. Take $r = \max(r_1, \dots, r_n)$. Hence $V_r(\infty) \subset U_i$ for all i.

Case 3. $c = -\infty$. Then for each i there exists $r_i \in \mathbb{R}$ such that $V_{r_i}(-\infty) \subset U_i$. Take $r = \min(r_1, \dots, r_n)$. Hence $V_r(-\infty) \subset U_i$ for all i.

(iii): Consider $c = 0$ and $U_n = (-\frac{1}{n}, \frac{1}{n})$, then $\cap_n U_n = \{0\}$ is not an open neighborhood of 0.

Q2. Let $f : A \to \mathbb{R}$ and $c \in A^c \subset [-\infty, \infty]$ and and $\ell \in [-\infty, \infty]$. Show that the definitions(given in Bartle, Lectures, Tutorials) for $\lim_{\substack{x \to c \\ x \in A}} f(x)$ are consistent with the following: $\lim_{\substack{x \to c \\ x \in A}} f(x) = \ell$ iff for any neighborhood U of ℓ, there exists neighborhood W of c such that the image

$$\{f(w) : w \in W \cap (A \setminus \{c\})\} \subset U$$

Solution. ($\Rightarrow$) **Case 1.** $\ell \in \mathbb{R}$. Suppose $\lim_{\substack{x \to c \\ x \in A}} f(x) = \ell$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$ we have $|f(x) - \ell| < \epsilon$. Let U be a neighborhood of ℓ. Then there exists $\epsilon > 0$ such that $V_\epsilon(\ell) \subset U$. Let $\delta > 0$ be chosen as in the above, and $W = V_\delta(c)$. Then $f(W \cap (A \setminus \{c\})) \subset U$ because for all $x \in W \cap (A \setminus \{c\}$, by definition $x \in A$ and $0 < |x - c| < \delta$.

Case 2. $\ell = \infty$. Let U be a neighborhood of ℓ. Then there exists M such that $V_M(\infty) \subset U$. By definition, there exists l such that for all $x > l$, $f(x) > M$. Take $W = V_l(\infty)$, then $f(W \cap A) \subset V_M(\infty) \subset U$. The case for $\ell = -\infty$ is similar.

($\Leftarrow$) **Case 1**. $\ell \in \mathbb{R}$. Let $\epsilon > 0$. Put $U = V_\epsilon(\ell)$. Then there exists neighborhood W of c such that $f(W \cap (A \setminus \{c\}) \subset U$. Then there exists $\delta > 0$ such that $V_\delta(c) \subset W$. Hence it follows that for all $x \in A$ with $0 < |x - c| < \delta$, $x \in W \cap (A \setminus \{c\})$ and therefore $|f(x) - \ell| < \epsilon$.

Case 2. $\ell = \infty$. Let $M > 0$ Put $U = V_M(\infty)$. By assumption there exists neighborhood W of ∞ such that $f(W \cap A) \subset V_M(\infty)$. Then there is l such that $V_l(\infty) \subset W$. Hence similar as in above, for all $x > l$, $x \in A$, we must have $f(x) > M$. The case for $\ell = -\infty$ is similar.

Q3. Let $f, g : X \to (0, \infty)$ and $x_0 \in X^c \cap \mathbb{R}$. Show that

[1] please send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

(i) $\lim_{\substack{x\to x_0\\x\in X}} f(x)=\ell\in[-\infty,\infty]$ iff $\lim_{\substack{x\to x_0\\x\in X}} f(x)=-\ell$.

(ii) $\lim_{\substack{x\to x_0\\x\in X}} g(x)=0$ iff $\lim_{\substack{x\to x_0\\x\in X}} \frac{1}{g(x)}=\infty$ (Can the assumption $g(x)\in(0,\infty)$ for all x replaced by $g(x)\in\mathbb{R}\setminus\{0\}$ for all x?)

(iii) If $\lim_{\substack{x\to x_0\\x\in X}} f(x)=\ell\in(0,\infty)$ and $\lim_{\substack{x\to x_0\\x\in X}} g(x)=0$, then $\lim_{\substack{x\to x_0\\x\in X}} \frac{f(x)}{g(x)}=+\infty$.

(iv) Show that $\lim_{\substack{x\to x_0\\x\in X}} \frac{x}{x\sqrt{x}}=+\infty$ by two methods:

(a) Use the results (i)-(iii).

(b) Check from def.

Solution. (i): $\lim_{\substack{x\to x_0\\x\in X}} f(x)=\ell$ iff: for all neighborhood U of ℓ there exists a neighborhood W of x_0 such that $f(W\cap(X\setminus\{x_0\}))\subset U$, i.e. $-f(W\cap(X\setminus\{x_0\}))\subset -U$. Note that for any neighborhood U of ℓ, $-U$ is a neighborhood of $-\ell$. Hence (i) follows.

(ii): $(\Rightarrow)$ Suppose $\lim_{\substack{x\to x_0\\x\in X}} g(x)=0$. Let $M>0$, then $\epsilon:=\frac{1}{M}>0$, there exists $\delta>0$ such that for all $x\in X$ with $x\in V_\delta(x_0)\setminus\{x_0\}$, $0\le g(x)<\epsilon=\frac{1}{M}$, so that $\frac{1}{g(x)}>M$.

$(\Leftarrow)$ Suppose $\lim_{\substack{x\to x_0\\x\in X}} \frac{1}{g(x)}=\infty$. Let $\epsilon>0$, then $M:=\frac{1}{\epsilon}>0$, so there exists $\delta>0$ such that for $x\in X$ with $x\in V_\delta(x_0)\setminus\{x_0\}$, we have $\frac{1}{g(x)}>M$, i.e. $0\le g(x)<\epsilon$.

The assumption that g is positively valued cannot be removed. To see this, take $g:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$ to be $g(x)=x$. Then $\lim_{\substack{x\to 0\\x\in X}} g(x)=0$ but $\lim_{\substack{x\to 0\\x\in X}} \frac{1}{g(x)}$ does not exist.

(iii): First notice that as $\lim_{\substack{x\to x_0\\x\in X}} f(x)=\ell>0$, there exists $\delta'>0$ such that $f(x)>\ell/2$ for $x\in V_\delta(x_0)\setminus\{x_0\}$. Let $M>0$. For the positive number $M\frac{2}{\ell}$, there exists $\delta''>0$ such that $\frac{1}{g(x)}>M\frac{2}{\ell}$, so that $\frac{f(x)}{g(x)}>M$.

(iv): (a): Take $X=(1,\infty)$, $f(x)=x$, $g(x)=x-\sqrt{x}$, then $\lim_{\substack{x\to 1\\x>1}} f(x)=1$, $\lim_{\substack{x\to 1\\x>1}} g(x)=0$. By (iii), $\lim_{\substack{x\to 1\\x>1}} \frac{f(x)}{g(x)}=+\infty$.

(b): Let $M>0$. Observe that if $0<x-1<\delta$, we have $1<x<\delta+1$, so $1<\sqrt{x}<\sqrt{\delta+1}$ and

$$\frac{x}{x-\sqrt{x}}>\frac{1}{\delta}$$

We want to find $\delta>0$ such that $\frac{1}{\delta}\ge M$. To do this we just have to choose $\delta=\frac{1}{M}$.

Q4. Can the assumption $x_0\in X^c\cap\mathbb{R}$ in **Q3** related to $x_0\in X^c$?

Solution. For (i), (ii), (iii), yes. The proofs are the same as in **Q3.**(Replace $V_\delta(x_0)$ by $V_l(\infty)$).

(Q3, 4, 5, 9, 13 of Bartle 4.3)

Q3. Let $f(x)=|x|^{1/2}$ for $x\neq 0$. Show that $\lim_{0\to 0+} f(x)=\lim_{x\to 0-} f(x)=+\infty$.

Solution. Let $M>0$, take $\delta=\frac{1}{M^2}$. Then for x with $0<x<\delta=\frac{1}{M^2}$, we have $\frac{1}{x^{1/2}}>M$. Hence $\lim_{x\to 0+} f(x)=\infty$. Because $f(x)=f(-x)$, so $\lim_{x\to 0-} f(x)=\infty$ as well.

Q5. Evaluate the following limits or show that they do not exist.

(a) $\lim_{x\to 1+} \frac{x}{x-1}$ $(x \neq 1)$

(b) $\lim_{x\to 1} \frac{x}{x-1}$ $(x \neq 1)$

(c) $\lim_{x\to 0+}(x+2)/\sqrt{x}$ $(x > 0)$

Solution. (a): $\lim_{x\to 1+} \frac{1}{x-1} = +\infty$ so $\lim_{x\to 1+} \frac{x}{x-1} = +\infty$ by **Q3**.

(b): Note $\lim_{x\to 1-} \frac{x}{x-1} = -\infty$, hence the limit cannot exist.

(c): $\lim_{x\to 0+} \frac{1}{\sqrt{x}} = +\infty$ and $\lim_{x\to 0+} x + 2 = 2$ implies that $\lim_{x\to 0+}(x+2)/\sqrt{2} = +\infty$ by **Q3**.

Q9. Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

Solution. Let $\epsilon > 0$. Notice $\lim_{x\to\infty} \frac{1}{x} = 0$, there exists l' such that $|\frac{1}{x}| < \epsilon$ for $x \in V_{l'}(\infty)$. By assumption, there exists l'' such that $|xf(x)| < |L| + 1$ for $x \in V_{l''}(\infty)$. Take $l = \max(l', l'')$, then $|f(x)| < (|L| + 1)\epsilon$.

Q13. Let f and g be defined on (a, ∞) and suppose $\lim_{x\to\infty} f = L$ and $\lim_{x\to\infty} g = \infty$. Prove that $\lim_{x\to\infty} f \circ g = L$.

Solution. Let U be a neighborhood of L. Then there exists a neighborhood W of ∞ such that $f(y) \in U$ for $y \in W$. For this neighborhood W, there exists a neighborhood V of ∞ such that $g(x) \in W$ for all $x \in V$. It follows that for all $x \in V$, $f \circ g(x) \in U$.