# MATH2050B 2021 HW 6

TA's solutions[1] to selected problems

**Q1.** Let $f : \mathbb{R} \to \mathbb{R}$ be additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that

(i) $f(0) = 0$

(ii) $f(-x) = -f(x)$

(iii) $f(nx) = nf(x)$ for all $n \in \mathbb{Z}, x \in \mathbb{R}$

(iv) $f(\frac{x}{m}) = \frac{f(x)}{m}$ for all $m \in \mathbb{N}, x \in \mathbb{R}$

(v) $f(rx) = rf(x)$ for all $r \in \mathbb{Q}, x \in \mathbb{R}$

and that $\lim_{x\to x_0} f(x)$ exists in $\mathbb{R}$ for some $x_0 \in \mathbb{R}$ iff $\lim_{x\to c} f(x)$ exists for any $c \in \mathbb{R}$ (what then $\lim_{x\to 0} f(x)$ is?) Show further that (assuming $\lim_{x\to x_0} f(x)$ exists in $\mathbb{R}$ for some $x_0 \in \mathbb{R}$), with $k := f(1)$, $f(x) = kx$ for all $x \in \mathbb{R}$.

**Solution.** $(i)$: Put $x, y = 0$ into $f(x+y) = f(x) + f(y)$ gives $f(0) = 0$. $(ii)$: $0 = f(x-x) = f(x) + f(-x)$ gives $f(-x) = -f(x)$ for all $x$.

$(iii)$: Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. The case $n = 0$ is $(i)$. First we deal with the case $n > 0$. Note the case when $n = 1$ is clearly true. Suppose for some $n_0 > 0$ we have $f(n_0x) = n_0f(x)$, then $f((n_0+1)x) = f(x)+f(n_0x) = (n_0+1)f(x)$. By MI $f(nx) = nf(x)$ for all $n > 0$ and $x \in \mathbb{R}$. For the case $n < 0$, note by the previous case and $(ii)$: $f(nx) = f(-n \cdot -x) = -nf(-x) = nf(x)$.

$(iv)$: Let $m \in \mathbb{N}, x \in \mathbb{R}$. By $(iii)$ $f(\frac{x}{m}) = \frac{1}{m}(mf(\frac{x}{m})) = \frac{1}{m}f(x)$. $(v)$: Let $r \in \mathbb{Q}$, then there exist $n \in \mathbb{Z}$, $m \in \mathbb{N}$ such that $r = \frac{n}{m}$, so $f(rx) = f(\frac{nx}{m}) = \frac{1}{m}f(nx) = rf(x)$.

Next, assume that $\lim_{x\to x_0} f(x)$ exists at one point $x_0$, we prove that $\lim_{x\to c} f(x)$ exists at any point $c \in \mathbb{R}$. Put $L = \lim_{x\to x_0} f(x)$. Let $\epsilon > 0$, then there exists $\delta > 0$ such that for any $x$ with $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \epsilon$.

Note for any $x$ with $0 < |x - c| < \delta$, we have $0 < |(x - c + x_0) - x_0| < \delta$, therefore

$$|f(x) - f(c - x_0) - L| = |f(x - c + x_0) - L| < \epsilon$$

We conclude that $\lim_{x\to c} f(x)$ exists and equals $f(c - x_0) + L$. To calculate $\lim_{x\to 0} f(x)$, use $f(0 + \frac{1}{n}) = f(0) + \frac{1}{n}f(1) \to 0$ as $n \to \infty$.

Finally, assume $\lim_{x\to x_0} f(x)$ exists in $\mathbb{R}$ for some $x_0$, and $k = f(1)$. Let $x \in \mathbb{R}$, we show $f(x) = kx$. Choose a sequence of rational numbers $(r_n)_{n=1}^{\infty}$ such that $r_n \to x$. By assumption $f(r_n) \to f(x)$. By $(v)$, $f(r_n) = r_nk$. Hence $f(x) = \lim_n r_nk = kx$.

*(Q14-17 of Section 4.1, 4th edition)*

**Q14.** Let $c \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to c} f(x)^2 = L$.

(a) Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$.

(b) Show by example that if $L \neq 0$, then $f$ may not have a limit at $c$.

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**Solution.** $(a)$: Let $\epsilon > 0$, then $\epsilon^2 > 0$, then there exists $\delta > 0$ such that for all $x$ with $0 < |x - c| < \delta$, we have $|f(x)^2| < \epsilon^2$, i.e. $|f(x)| < \epsilon$. Hence $\lim_{x \to c} f(x)$ exists and equals 0.

$(b)$: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. Then $\lim_{x \to 0} f(x)^2 = 1 = L$ but $\lim_{x \to 0} f(x)$ does not exist.

**Q15.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if $x$ is rational, and $f(x) = 0$ if $x$ is irrational.

(a) Show that $f$ has a limit at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then $f$ does not have a limit at $c$.

**Solution.** $(a)$: Let $\epsilon > 0$. Choose $\delta = \epsilon > 0$. For any $x$ with $0 < |x| < \delta$, we have either $f(x) = x$ or $f(x) = 0$. So $|f(x) - 0| < \epsilon$ and hence $\lim_{x \to 0} f(x)$ exists.

$(b)$: Choose a sequence $(r_n)$ of rational numbers with $r_n \to c$, and a sequence $(t_n)$ of irrational numbers with $t_n \to c$. Then $f(r_n) \to c$ and $f(t_n) \to 0$. Hence $\lim_{x \to c} f(x)$ does not exist.

**Q16.** Let $f : \mathbb{R} \to \mathbb{R}$, let $I$ be an open interval in $\mathbb{R}$ and let $c \in I$. If $f_1$ is the restriction of $f$ to $I$, show that $f_1$ has a limit at $c$ if and only if $f$ has a limit at $c$, and that the limits are equal.

**Solution.** ($\Rightarrow$)Assume $f_1$ has a limit at $c$, say the limit is $L$. Let $\epsilon > 0$, then there exists $\delta_1 > 0$ such that for $x \in I$ with $0 < |x - c| < \delta$, we have $|f_1(x) - L| < \epsilon$.

Choose $\delta < \delta_1$ such that $V_\delta(c) \subset I$ (this is do-able because $I$ is open), then for all $x \in \mathbb{R}$ with $0 < |x - c| < \delta$, we must have $x \in I$ and so $|f(x) - L| < \epsilon$. Hence $\lim_{x \to c} f(x) = L$.

($\Leftarrow$)Assume $f$ has a limit at $c$, say the limit is $L$. Let $\epsilon > 0$, then there exists $\delta > 0$ such that for $x \in \mathbb{R}$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. Now, for any $x \in I$ with $0 < |x - c| < \delta$, we have $|f_1(x) - L| < \epsilon$. Hence $\lim_{x \to c} f_1(x) = L$.

**Q17.** Let $f : \mathbb{R} \to \mathbb{R}$, let $J$ be a closed interval in $\mathbb{R}$, and let $c \in J$. If $f_2$ is the restriction of $f$ to $J$, show that if $f$ has a limit at $c$ then $f_2$ has a limit at $c$. Show by example that it does not follow that if $f_2$ has a limit at $c$ then $f$ has a limit at $c$.

**Solution.** The first part is identical to ($\Leftarrow$) part of **Q16**. Consider the function $f$ defined in **Q14(b)**, $J = [0, 1]$, then $f_2 : [0, 1] \to \mathbb{R}$ is given by $f_2(x) = 1$, and clearly $\lim_{x \to 0} f_2(x) = 1$. But $\lim_{x \to 0} f(x)$ does not exist.

**Q3.** Use $\epsilon$-$\delta$ definition to check that

(i) $\lim_{x \to -1} \frac{x+5}{2x+3} = 4$

(ii) $\lim_{x \to 0} x + \text{sgn}(x)$, $\lim_{x \to 0} \sin(\frac{1}{x^2})$ does not exist in $\mathbb{R}$

**Solution**. $(i)$ Note $\frac{x+5}{2x+3} - 4 = (x+1)\frac{-7}{2x+3}$, and if $0 < |x + 1| < \frac{1}{10}$, then $\frac{4}{5} < 2x + 3 < \frac{6}{5}$. Let $\epsilon > 0$, take $\delta = \min(\epsilon, \frac{1}{10})$, for any $x$ with $0 < |x + 1| < \delta$, we have

$$|\frac{x+5}{2x+3} - 4| < \epsilon \frac{35}{4}$$

It follows that $\lim_{x \to -1} \frac{x+5}{2x+3} = 4$.

$(ii)$: Suppose on the contrary that $\lim_{x\to 0} x + \operatorname{sgn}(x) = L$ exists. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for any $x$ with $0 < |x| < \delta$, $|x + \operatorname{sgn}(x) - L| < \epsilon$.

Note for all large $n$, $|\pm \frac{1}{n}| < \delta$ (this statement means: there exists $N$ such that $|\pm \frac{1}{n}| < \delta$ for all $n \geq N$) We have that

$$\Big|\frac{1}{n} + \operatorname{sgn}(\frac{1}{n}) - L\Big| < \epsilon, \text{ for all large } n$$

Taking $n \to \infty$, $|1 - L| \leq \epsilon$. Because $\epsilon$ is arbitrarily chosen, $1 = L$. On the other hand we can replace $\frac{1}{n}$ by $-\frac{1}{n}$ in the above inequality, which will give us $|-1 - L| < \epsilon$ for all $\epsilon$. Hence $L = -1$. Contradiction.

Next, suppose on the contrary that $\lim_{x\to 0} \sin(\frac{1}{x^2}) = L$ exists. Let $\frac{1}{2} > \epsilon > 0$, then there exists $\delta > 0$ such that for any $x$ with $0 < |x| < \delta$, we have

$$\Big|\sin(\frac{1}{x^2}) - L\Big| < \epsilon$$

Put $x_n = \frac{1}{\sqrt{2n\pi}}$ where $n \in \mathbb{N}$. For all large $n$, we have $0 < |x_n| < \delta$, $\sin(\frac{1}{x_n^2}) = 0$. On the other hand, if we put $y_n = \frac{1}{\sqrt{2n\pi + \pi/2}}$, then for all large $n$, $0 < |y_n| < \delta$, $\sin(\frac{1}{y_n^2}) = 1$. Now

$$1 = |0 - 1| = \Big|\sin(\frac{1}{x_n^2}) - \sin(\frac{1}{y_n^2})\Big| \leq \Big|\sin(\frac{1}{x_n^2}) - L\Big| + \Big|\sin(\frac{1}{y_n^2}) - L\Big| < 2\epsilon$$

But $\epsilon < \frac{1}{2}$ by assumption. Contradiction.

*(Q1, 3, 8-11, 15 of Section 4.2, 4th edition)*

**Q1.** Apply Theorem 4.2.4 to determine the following limits:

(a) $\lim_{x\to 1}(x+1)(2x+3)$

(b) $\lim_{x\to 1} \frac{x^2+2}{x^2-2}$

(c) $\lim_{x\to 2} \frac{1}{x+1} - \frac{1}{2x}$

(d) $\lim_{x\to 0} \frac{x+1}{x^2+2}$

**Solution.** (a): Note $\lim_{x\to 1} x + 1 = 2$ and $\lim_{x\to 1} 2x + 3 = 5$, so the required limit is 10.

(b): Note $\lim_{x\to 1} x^2 + 2 = 3$, $\lim_{x\to 1} x^2 - 2 = -1$, so the required limit is $-3$.

(c): Note $\lim_{x\to 2} \frac{1}{x+1} = \frac{1}{3}$ and $\lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$, so the required limit is $\frac{1}{12}$.

(d): Note $\lim_{x\to 0} x + 1 = 1$ and $\lim_{x\to 0} x^2 + 2 = 2$, so the required limit is $\frac{1}{2}$.

**Q3.** Find $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x > 0$.

**Solution.** Notice that

$$\frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2} = \frac{-1}{(1+2x)(\sqrt{1+2x} + \sqrt{1+3x})}$$

Because $\lim_{x\to 0} 1+2x = 1$, $\lim_{x\to 0} \sqrt{1+2x} + \sqrt{1+3x} = 2$, it follows from Theorem 4.2.4 that $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} = -\frac{1}{2}$.

**Q8.** Let $n \in \mathbb{N}$ be such that $n \geq 3$. Derive the inequality $-x^2 \leq x^n \leq x^2$ for $-1 < x < 1$. Then use the fact that $\lim_{x\to 0} x^2 = 0$ to show that $\lim_{x\to 0} x^n = 0$.

**Solution.** Let $-1 < x < 1$. Note $|x| < 1$, therefore $|x^n| < x^2$. Hence $\lim_{x\to 0} x^n = 0$ by squeeze theorem.

**Q9.** Let $f, g$ be defined on $A$ to $\mathbb{R}$ and let $c$ be a cluster point of $A$.

(a) Show that if both $\lim_{x\to c} f$ and $\lim_{x\to c} f+g$ exist, then $\lim_{x\to c} g$ exists.

(b) If $\lim_{x\to c} f$ and $\lim_{x\to c} fg$, does it follow that $\lim_{x\to c} g$ exists?

**Solution.** (a) follows from the addition rule and $g = f + g - f$. (b): Let $g : A \to \mathbb{R}$ be any function such that $g$ does not have a limit at $c$ (try to explicitly define one). Take $f : A \to \mathbb{R}$ be $f(x) = 0$. Then the assumptions are satisfied but $\lim_{x\to c} g$ does not exist.

**Q10.** Give examples of functions $f$ and $g$ such that $f$ and $g$ do not have limits at a point $c$, but such that both $f+g$ and $fg$ have limits at $c$.

**Solution.** Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Let $g[-1, 1] \to \mathbb{R}$ be $g(x) = -f(x)$. Then $f(x) + g(x) = 0$ for all $x$ and $f(x)g(x) = 1$ for all $x$. Hence $f$, $g$ are the desired functions.

**Q11.** Determine whether the following limits exist on $\mathbb{R}$.

(a) $\lim_{x\to 0} \sin(1/x^2)$ $(x \neq 0)$

(b) $\lim_{x\to 0} x\sin(1/x^2)$ $(x \neq 0)$

(c) $\lim_{x\to 0} \operatorname{sgn}\sin(1/x)$ $(x \neq 0)$

(d) $\lim_{x\to 0} \sqrt{x}\sin(1/x^2)$ $(x > 0)$

**Solution.** (a): Please refer to **Q3**. (b): limit exists and equals 0 because

$$\left|x\sin(\frac{1}{x^2})\right| \leq |x|$$

(c): Limit does not exist. This can be seen by using sequential criteria: Let $x_n = \frac{1}{2n\pi+\pi/2}$, $y_n = \frac{1}{2n\pi-\pi/2}$. Then $x_n, y_n \to 0$ but $\operatorname{sgn}\sin(1/x_n) = 1$ and $\operatorname{sgn}\sin(1/y_n) = -1$ for all $n$.

(d): limit exists and equals 0 because

$$\left|\sqrt{x}\sin(\frac{1}{x^2})\right| \leq |\sqrt{x}|$$

**Q15.** Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of $A$. In addition, suppose that $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) = \sqrt{f(x)}$. If $\lim_{x\to c} f(x)$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$.

**Solution.** Let $L = \lim_{x\to c} f(x)$.

**Case 1.** $L = 0$. Let $\epsilon > 0$, then $\epsilon^2 > 0$, and there exists $\delta > 0$ such that $|f(x)| < \epsilon^2$ for all $x \in A$ with $0 < |x - c| < \delta$. Hence $|\sqrt{f}(x)| < \epsilon$ for all $x \in A$ with $0 < |x - c| < \delta$.

**Case 2.** $L \neq 0$. Then $L > 0$. Let $\epsilon > 0$. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon\sqrt{L}$ for all $x \in A$ with $0 < |x - c| < \delta$:

$$|\sqrt{f(x)} - \sqrt{L}| = \frac{|f(x) - L|}{|\sqrt{f(x)} + \sqrt{L}|} < \frac{|f(x) - L|}{\sqrt{L}} < \epsilon$$