MATH2050B 2021 Assignment 2
TA's solutions[1] to selected problems

Q1. Using the axiom of $\mathbb{R}$, do:

(i) If A is a subset of $\mathbb{R}$, bounded below but not above, and satisfies the property: $x \in A$ whenever there exists $a_1, a_2 \in A$ such that $a_1 < x < a_2$. Show that A is an interval

(ii) State and prove the nested interval theorem.

Solution. (i): Let $a = \inf(A) \in \mathbb{R}$. We prove that $A = (a, \infty)$ or $A = [a, \infty)$:

Case 1. Suppose $a \in A$. Let $x \in A$, then $x \geq a$, thus $A \subset [a, \infty)$. Conversely let $x \in [a, \infty)$. if $x = a$ then certainly $x \in A$. If $x \neq a$, because A is not bounded above we can find $a_2 \in A$ such that $x < y$. Now we have $a < x < a_2$, so by the assumption on A, $x \in A$. Hence $A = [a, \infty)$.

Case 2. Suppose $a \notin A$. Let $x \in A$, then $x > a$, thus $A \subset (a, \infty)$. Conversely let $x \in (a, \infty)$. Because $x > a$, by definition of infimum, there exists $a_1 \in A$ such that $a < a_1 < x$. On the other hand, A is not bounded above means that there exists $a_2 \in A$ such that $x < a_2$. Now $a_1 < x < a_2$ where $a_1, a_2 \in A$. By the assumption on A, $x \in A$. Hence $A = (a, \infty)$.

(ii): (Nested interval theorem) Let $I_n = [a_n, b_n]$ be a sequence of intervals so that $I_{k+1} \subset I_k$ for all $k = 1, 2, \ldots$, then $\cap_{k=1}^{\infty} I_k \neq \varnothing$.

Proof. By the nested property, $[a_n, b_n] = I_n \subset I_1$ for all $n = 1, 2, \ldots$, so $a_n \leq b_1$ for all n. This shows that the set $\{a_n : n = 1, 2 \ldots\}$ is bounded above, let $a = \sup(A)$. We claim that $a \in \cap_{k=1}^{\infty} I_k$. To do this we need to show $a_n \leq a \leq b_n$ for all n. $a_n \leq a$ for all n is clear, it remains to show $a \leq b_n$ for all n. Suppose on the contrary that $a > b_k$ for some k. Then for all $n \geq k$, $a_n \leq b_k < a$. By the nested property, a_n is an increasing sequence, so we have in fact $a_n \leq b_k < a$ for all $n = 1, 2, \ldots$, this gives $a \leq b_k < a$ which is a contradiction.

Q2. For each of the following, compute the limit if exists in $\mathbb{R}^*$ or show that the limit does not exist. Check each of your answers by definition.

(i) $\lim_{x \to 7} \frac{x}{x-7}$, $x < 7$

(ii) $\lim_{x \to 1} \frac{x^2}{x^2-2}$, $x \neq \sqrt{2}$

(iii) $\lim_{x \to \sqrt{2}} \frac{x^2}{x^2-2}$, $x \neq \sqrt{2}$

(iv) $\lim_{x \to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$, $x > 0$

Solution. (i): The limit is $-\infty$. Let $M < 0$. For $x < 7$ with $0 < 7 - x < \delta$ where $\delta := \min(6, -\frac{1}{M})$, we have $x > 1$ and $7 - x < -\frac{1}{M}$. It follows that $\frac{x}{x-7} < -\frac{1}{\delta} \leq M$.

(ii): The limit is -1. Note $|\frac{x^2}{x^2-2} + 1| = 2\frac{x^2-1}{x^2-2}$. First observe that if $0 < |x - 1| < \frac{1}{10}$, then $\frac{9}{10} < x < \frac{11}{10}$ and so

$$-\frac{100}{119} > \frac{1}{x^2 - 2} > -\frac{100}{79}$$

[1] please send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Let $\epsilon > 0$, because $\lim_{x\to 1} x^2 = 1$ (see proof below), there exists δ_0 so that for all x with $0 < |x-1| < \delta_0$, $|x^2 - 1| < \epsilon$. Take $\delta = \min(\delta_0, \frac{1}{10})$, then for x with $0 < |x-1| < \delta$, we have

$$|\frac{x^2}{x^2-2} + 1| = 2|\frac{x^2-1}{x^2-2}| < \frac{200}{79}\epsilon$$

Proof of ($\lim_{x\to 1} x^2 = 1$). Let $\epsilon > 0$. Note that if $0 < |x-1| < \frac{1}{2}$, then $\frac{3}{2} < x+1$. Take $\delta = \min(\epsilon, \frac{1}{2})$, then for all x with $0 < |x-1| < \delta$, we have $|x^2-1| = |x-1|\,|x+1| < \frac{3}{2}\epsilon$.

(iii). The limit does not exist. Suppose on the contrary that the limit exists and equal to $\ell \in [-\infty, \infty]$. Then $\lim_{x\to\sqrt{2}+} \frac{x^2}{x^2-2} = \ell = \lim_{x\to\sqrt{2}-} \frac{x^2}{x^2-2}$ too.

Claim. $\lim_{x\to\sqrt{2}+} \frac{x^2}{x^2-2} = +\infty$. Let $M > 0$. Take $\delta = \min(\frac{1}{M}, \frac{1}{10})$, then for x with $0 < x - \sqrt{2} < \delta$, we have $\frac{1}{x+\sqrt{2}} > (\frac{1}{10} + 2\sqrt{2})^{-1}$ and $\frac{1}{x-\sqrt{2}} > \frac{1}{\delta} = M$, so

$$\frac{x^2}{x^2-2} \geq \frac{2}{x^2-2} \geq M(\frac{1}{10} + 2\sqrt{2})^{-1}$$

Because $(\frac{1}{10} + 2\sqrt{2})^{-1}$ is a constant, it follows that $\lim_{x\to\sqrt{2}+} \frac{x^2}{x^2-2} = +\infty$.

Now, by our assumption that $\lim_{x\to\sqrt{2}-} \frac{x^2}{x^2-2} = \ell = +\infty$, we can conclude a contradiction. This is because the function $x \mapsto \frac{x^2}{x^2-2}$ takes values in $(-\infty, 0)$ for $x < \sqrt{2}$. It is impossible that $\lim_{x\to\sqrt{2}-} \frac{x^2}{x^2-2} \in (0, \infty]$.

(iv): The limit is $-\frac{1}{2}$. We first split the problem to several easier subproblems. Note

$$\frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2} + \frac{1}{2} = \frac{\sqrt{1+2x} + \sqrt{1+3x} - 2}{(1+2x)(\sqrt{1+2x} + \sqrt{1+3x})} + \frac{2x}{1+2x}$$

The problem is solved if we can show:

(A) $\lim_{x\to 0+} \frac{\sqrt{1+2x}-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} = 0$

(B) $\lim_{x\to 0+} \frac{\sqrt{1+3x}-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} = 0$

(C) $\lim_{x\to 0+} \frac{2x}{1+2x} = 0$

(A): Let $\epsilon > 0$. Take $\delta = \epsilon$. Note that $0 < \sqrt{1+2x} - 1 \leq 2x$ for $x > 0$. It follows that

$$0 \leq \frac{\sqrt{1+2x} - 1}{(1+2x)(\sqrt{1+2x} + \sqrt{1+3x})} \leq \frac{2x}{2} < \epsilon$$

for x with $0 < x < \delta$.

(B): is the same with (A). Observe that $0 < \sqrt{1+3x} - 1 \leq 3x$ for $x > 0$.

(C): Let $\epsilon > 0$. Take $\delta = \epsilon$. Then for x with $0 < x < \delta$, we have $0 \leq \frac{2x}{1+2x} \leq \frac{2x}{2} < \epsilon$.

Q3. Let $N \geq 2$ be a natural number and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^N$ for all $x \in \mathbb{R}$. Show by definition that f is continuous but not uniformly continuous.

Solution. (f is continuous): Pick any $x_0 \in \mathbb{R}$. We show that f is continuous at x_0. Note on $[x_0 - 1, x_0 + 1]$, $|x|$ is bounded by some M. Let $\epsilon > 0$. Take $\delta = \min(\epsilon, 1)$, then for x with $|x - x_0| < \delta$, we have $x \in [x_0 - 1, x_0 + 1]$, so $|x| \leq M$, so by the assumption $N \geq 2$:

$$|x^N - x_0^N| = |x - x_0|\,|x^{N-1} + x^{N-2}x_0 + \cdots + x_0^{N-1}|$$

By triangle inequality:

$$|x^N - x_0^N| \leq |x - x_0|(NM^{N-1}) \leq \epsilon(NM^{N-1})$$

Hence f is continuous on $\mathbb{R}$.

(f is not uniformly continuous on $\mathbb{R}$): Suppose on the contrary that f is uniform continuous on $\mathbb{R}$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|x^N - y^N| < 1$.

For $n \in \mathbb{N}$, put $x_n = n + \delta/2$, $y_n = n$. It is clear that $|x_n - y_n| < \delta$ for all n, we will prove that there exists n such that $|x_n^N - y_n^N| > 1$ which will be a contradiction.

By binomial theorem, $x_n^N \geq n^N + n^{N-1}\frac{\delta}{2}$, so $x_n^N - y_n^N \geq n^{N-1}\frac{\delta}{2}$. The RHS $\to \infty$ as $n \to \infty$, so there exists n such that $|x_n^N - y_n^N| = x_n^N - y_n^N > 1$.

Q4. Let $\varnothing \neq A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$. By virtue of definition, prove or disprove for each of the following assertions

(i) If f, g are continuous then so is fg.

(ii) If f, g are uniformly continuous then so is fg.

Answer the same question again if $A = (0, 1)$.

Solution. (i) The assertion is true. For $\epsilon_0 := 1$, there exists $\delta' > 0$ such that for x with $|x - x_0| < \delta'$, we have $|f(x) - f(x_0)| < 1$, so $f(x_0) - 1 < f(x) < f(x_0) + 1$. Therefore $f(x)$ is bounded by some $M > 0$ on $V_{\delta''}(x_0)$.

Pick any $x_0 \in A$. Let $\epsilon > 0$. Then there exists $\delta'' > 0$ such that for x with $|x - x_0| < \delta''$, we have $|f(x) - f(x_0)| < \epsilon/2M$ and $|g(x) - g(x_0)| < \epsilon/2$.

Take $\delta = \min(\delta', \delta'')$. For $x \in A$ with $|x - x_0| < \delta$, we have

$$|f(x)g(x) - f(x_0)g(x_0)| \leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| < \epsilon$$

(ii). The assertion is not true. Take $A = \mathbb{R}$, $f(x) = g(x) = x$. Then Q3 showed that fg is not uniformly continuous continuous on $\mathbb{R}$. But it is clear that f, g are uniformly continuous on $\mathbb{R}$.

If $A = (0, 1)$, then (i) is still true. We will prove that in this case (ii) is also true.

Consider $\epsilon_0 := 1$. There exists $\delta > 0$ such that for $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < 1$ and $|g(x) - g(y)| < 1$.

Because A is bounded, there are finitely many $x_1, \ldots, x_N \in A$ such that

$$\bigcup_{i=1}^{N} V_\delta(x_i) = \bigcup_{i=1}^{N} (x_i - \delta, x_i + \delta) \supset (0, 1)$$

On each $V_\delta(x_i)$, f, g are bounded by some $M_i > 0$, i.e. $|f(x)| < M_i$ and $|g(x)| < M_i$ for all $x \in V_\delta(x_i)$. (Take $M_i = |f(x_i)| + 1$, then for all $x \in V_\delta(x_i)$, $|f(x)| - |f(x_i)| \leq |f(x) - f(x_i)| < 1$) Let $M = \max(M_i)$, we have f, g are bounded by M on A.

Let $\epsilon > 0$. By uniform continuity of f, g, there exists $\delta' > 0$ such that for all $x, y \in A$ with $|x - y| < \delta'$, $|f(x) - f(y)| < \epsilon/2M$ and $|g(x) - g(y)| < \epsilon/2M$.

For $x, y \in A$ with $|x - y| < \delta'$, we have

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| < \epsilon$$

Hence fg are uniformly continuous on A.