MATH2050B 2021 HW 4
TA's solutions[1] to selected problems

Q1. Let $\varnothing \neq A \subset \mathbb{R}$, bounded with $x := \sup A \in \mathbb{R}$. Show that there exists a sequence (x_n) in A such that $\lim_n x_n = x$. Moreover, if $x \notin A$ show that you can have your (x_n) satisfying additionally that $x_n < x_{n+1}$ for all n.

Solution. If $x \in A$, then we can take $x_n = x$ for all n.

If $x \notin A$, we construct (x_n) such that (x_n) strictly increases to x.

Let $\epsilon_1 = 1$. Then there exists $x_1 \in A$ such that $x - \epsilon_1 < x_1 < x$.

Because $x \notin A$, so $x_1 \neq x$. Let $\epsilon_2 = \min(x - x_1, 1/2)$. Note $\epsilon_2 > 0$, therefore there exists $x_2 \in A$ such that $x - \epsilon_2 < x_2 < x$. Notice that $x_1 < x_2$ and $x - 1/2 < x_2 < x$.

Now let $\epsilon_3 = \min(x - x_2, 1/3)$. Then by the argument above there exists $x_3 \in A$, $x_2 < x_3$ and $x - \frac{1}{2} < x_3 < x$.

Let $\epsilon_4 = \min(x - x_3, 1/4)$, inductively we can find a sequence (x_n) such that $x_1 < x_2 < \dots$ and $x - \frac{1}{n} < x_n < x$. Hence this is the desired sequence.

Q2. Let (a_n) be a bounded sequence, and

$$t_n = \inf\{a_m : m \geq n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\},$$

$$s_n = \sup\{a_m : m \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Show that $(t_n), (s_n)$ are monotone and

$$\lim_n t_n = \sup\{t_n : n \in \mathbb{N}\} \leq \inf\{s_k : k \in \mathbb{N}\} = \lim_k s_k.$$

Solution. For each n, $t_n \leq a_m$ for all $m \geq n$. In particular, $t_n \leq a_m$ for all $m \geq n+1$, and so

$$t_n \leq \inf\{a_m : m \geq n+1\} = t_{n+1}.$$

This shows that (t_n) is increasing.

For each n, $s_n \geq a_m$ for all $m \geq n$. In particular, $s_n \geq a_m$ for all $m \geq n+1$, and so

$$s_n \geq \sup\{a_m : m \geq n+1\} = s_{n+1}.$$

This shows that (s_n) is decreasing.

Since (a_n) is bounded, so (t_n) and (s_n) are also bounded. By MCT, $\lim_n t_n$ exists, $\lim_n t_n = \sup\{t_n : n \in \mathbb{N}\}$, and $\lim_n s_n$ exists, $\lim_n s_n = \inf\{s_n : n \in \mathbb{N}\}$. Because $t_n \leq s_n$ for all n, it follows also $\lim_n t_n \leq \lim_n s_n$.

Q3. Let $(a_n), (t_n), (s_n)$ be as in **Q2**. Show that (a_n) converges iff $\lim_n t_n = \lim_n s_n$.

($\lim_n t_n$ is usually denoted by $\liminf_n a_n$. $\lim_n s_n$ is usually denoted by $\limsup_n a_n$.)

Solution. ($\Rightarrow$)Suppose that (a_n) is convergent to $a \in \mathbb{R}$. We claim that $\lim_n t_n = \lim_n s_n = a$.

[1]please kindly send an email to `ncliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Let $\epsilon > 0$, then there is N so that $|a_n - a| < \epsilon$ for all $n \geq N$. This is to say for all $n \geq N$,

$$a - \epsilon < a_n < a + \epsilon.$$

It follows that for all $n \geq N$, $a - \epsilon \leq t_n \leq a + \epsilon$ and $a - \epsilon \leq s_n \leq a + \epsilon$. This implies that for all $n \geq N$, $|t_n - a| \leq \epsilon$ and $|s_n - a| \leq \epsilon$, which shows the claim.

($\Leftarrow$) Suppose that $\lim_n t_n = \lim_n s_n = a \in \mathbb{R}$. We claim that a_n converges to a. Let $\epsilon > 0$. Then there is N so that for all $n \geq N$,

$$a - \epsilon < t_n < a + \epsilon$$

and

$$a - \epsilon < s_n < a + \epsilon.$$

This two conditions imply that for all $n \geq N$, $a - \epsilon < a_n < a + \epsilon$. Hence $\lim_n a_n$ exists and equals a.