§5.5 Argument Principle, Rouché's Theorem, Open Mapping Theorem and Hurwitz Theorem

Def: A function $f: D \to \mathbb{C}$ is said to be meromorphic if $\forall z \in D$, f is analytic or has pole at z.

Thm 1: Let C be a positively oriented simple closed contour, f be a meromorphic function inside and on C such that f has no zero or pole on C. Let $a_1, \cdots, a_n$ be zeros of f inside C with order $\alpha_1, \cdots, \alpha_n$, resp; $b_1, \cdots, b_m$ be poles of f inside C with order $\beta_1, \cdots, \beta_m$, resp. Then $\forall$ function $\varphi(z)$ analytic inside and on C, we have

$$\boxed{\frac{1}{2\pi i}\int_C \varphi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n}\alpha_k\varphi(a_k) - \sum_{j=1}^{m}\beta_j\varphi(b_j)}.$$

Pf: Note that by assumption, f has only finitely many zeros and poles inside C. Therefore, the function $F(z) = \varphi(z)\frac{f'(z)}{f(z)}$ is analytic inside and on $\mathbb{C}$ except finitely many isolated singular points at $a_1, \cdots, a_n$ and $b_1, \cdots, b_m$.

Cauchy Integral Famula implies

$$\frac{1}{2\pi i}\int_C \varphi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} F(z) + \sum_{j=1}^{m} \operatorname{Res}_{z=b_j} F(z).$$

Note that in a small ε-nbd of a_k,

$$f(z) = (z-a_k)^{\alpha_k} g(z)$$

for some analytic $g(z)$ with $g(a_k) \neq 0$.

Hence

$$\varphi(z)\frac{f'(z)}{f(z)} = \varphi(z)\left(\frac{\alpha_k}{z-a_k} + \frac{g'(z)}{g(z)}\right).$$

$$= \frac{\alpha_k \varphi(z)}{z-a_k} + \varphi(z)\frac{g'(z)}{g(z)}$$

Since $\varphi(z)\dfrac{g'(z)}{g(z)}$ is analytic, we have

$$\operatorname{Res}_{z=a_k} F(z) = \begin{cases} \alpha_k \varphi(a_k) & \text{if } \varphi(a_k) \neq 0 \\ 0 & \text{if } \varphi(a_k) = 0 \end{cases}$$

$$= \alpha_k \varphi(a_k) \quad (\text{in both cases})$$

Similarly, in a small ε-nbd. of b_j,

$$f(z) = \frac{h(z)}{(z-b_j)^{\beta_j}}$$

for some analytic $h(z)$ with $h(b_j) \neq 0$.

Hence

$$\varphi(z)\frac{f'(z)}{f(z)} = \varphi(z)\left(\frac{-\beta_j}{z-b_j} + \frac{h'(z)}{h(z)}\right)$$

$$= \frac{-\beta_j \varphi(z)}{z-b_j} + \varphi(z)\frac{h'(z)}{h(z)}$$

$$\Rightarrow \quad \operatorname{Res}_{z=b_j} F(z) = -\beta_j \varphi(b_j)$$

$$\therefore \quad \frac{1}{2\pi i}\int_C \varphi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \alpha_k \varphi(a_k) - \sum_{j=1}^{m} \beta_j \varphi(b_j)$$

<u>Cor 1</u> Under the same assumptions as in Thm 1, we have

$$\boxed{\frac{1}{2\pi i}\int_C z^{\ell}\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \alpha_k a_k^{\ell} - \sum_{j=1}^{m} \beta_j b_j^{\ell}}$$

$\forall\, \ell = 0, 1, 2, \cdots$.

(<u>Pf</u> : $\varphi(z) = z^{\ell}$ is entire.)

Thm 2 (Argument Principle) Let C be a positively oriented simple closed contour, f be a meromorphic function inside and on C such that f has no zero or pole on C. Then

$$\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_0(f) - N_p(f)}$$

where $N_0(f)$ & $N_p(f)$ = number of zeros & poles, respectively, of f inside C (counting multiplicities).

Pf: Take $l=0$ in Cor 1, we have

$$\frac{1}{2\pi i}\int_C 1\cdot\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n}\alpha_k\cdot 1 - \sum_{j=1}^{m}\beta_j\cdot 1$$

$$= N_0(f) - N_p(f)$$

(counting multiplicities) ※

Remark: There is a topological interpretation of the term $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$ (which is an integer by Thm 2)

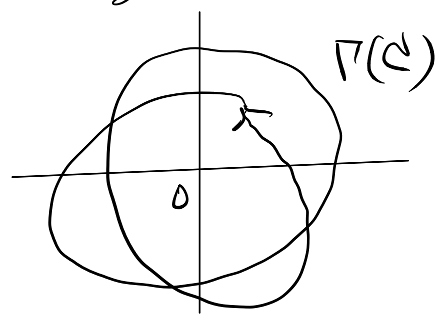
In fact, if C is (smoothly) parametrized by

$z(t)$ for $t \in [a,b]$, then $\Gamma = f(C)$ is a contour parametrized by $\zeta(t) = f(z(t))$, $t \in [a,b]$.

By assumption $|\zeta(t)| \neq 0$ (and $|\zeta(t)| \neq \infty$), $\forall t \in [a,b]$.

(i.e. the contour Γ never touch the origin.)

Then $\dfrac{\zeta(t)}{|\zeta(t)|}$ is a <u>smooth</u> function on $a \le t \le b$.



We <u>claim</u> that there is a <u>smooth</u> function $\theta(t)$ on $a \le t \le b$ such that $\dfrac{\zeta(t)}{|\zeta(t)|} = e^{i\theta(t)}$, $a \le t \le b$.

If this is true, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))z'(t)}{f(z(t))}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{\zeta'(t)}{\zeta(t)}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{\left[|\zeta(t)|e^{i\theta(t)}\right]'}{|\zeta(t)|e^{i\theta(t)}}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \left[\frac{\frac{d}{dt}|\zeta(t)|}{|\zeta(t)|} + i\theta'(t)\right]dt$$

$$= \frac{1}{2\pi i}\left\{\left[\ln|\zeta(t)|\right]_a^b + i\left[\theta(t)\right]_a^b\right\}$$

$$= \frac{1}{2\pi}\left[\theta(b)-\theta(a)\right] \quad (\text{since } \zeta(b)=\zeta(a))$$

Note that $\zeta(t) = |\zeta(t)|e^{i\theta(t)}$

$$\Leftrightarrow \quad f(z(t)) = |f(z(t))|e^{i\theta(t)}$$

i.e. $\theta(t) \in \arg f(z(t))$

And hence $\theta(b) \in \arg f(z(b)) = \arg f(z(a)) \ni \theta(a)$

$$\Rightarrow \quad \theta(b)-\theta(a) = 2k\pi \text{ for some } k\in\mathbb{Z}.$$

Abusing the notation, we usually denote $\theta(b)-\theta(a)$ by $\Delta_C \arg f(z)$, i.e. <u>change of argument of $f(z)$ along C</u>. Therefore

$$\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\Delta_C \arg f(z)}$$

and hence the name "Argument principle" for Thm 2.

Pf of claim : It is equivalent to show that if $(u(t), v(t))$, $a \le t \le b$, with $u^2+v^2 \equiv 1$, there exists $\theta(t)$ such that

$$u(t) = \cos\theta(t) \quad \& \quad v(t) = \sin\theta(t).$$

Note that one can always find θ_0 such that

$$u(a) = \cos\theta_0 \quad \& \quad v(a) = \sin\theta_0 .$$

Then the following function

$$\theta(t) = \theta_0 + \int_a^t (uv' - vu')\,dt$$

is the required function. To see this, we note that

$$\text{and} \quad u^2+v^2 \equiv 1, \quad \begin{cases} \theta' = uv' - vu' \\ 0 = uu' + vv' \end{cases}$$

$$\underset{(\text{using } u^2+v^2=1)}{\overset{(\text{solve})}{\Longrightarrow}} \quad \begin{cases} u' = -v\theta' \\ v' = u\theta' \end{cases}$$

Then $\frac{d}{dt}\left\{\frac{1}{2}\left[(u-\cos\theta)^2 + (v-\sin\theta)^2\right]\right\}$

$$= (u-\cos\theta)(u'+\theta'\sin\theta) + (v-\sin\theta)(v'-\theta'\cos\theta)$$

$$= (u-\cos\theta)(-v+\sin\theta)\theta' + (v-\sin\theta)(u-\cos\theta)\theta'$$

$$= 0$$

$\therefore$ $(u-\cos\theta)^2+(v-\sin\theta)^2$ is a constant function and hence $(u-\cos\theta)^2+(v-\sin\theta)^2 = (u(a)-\cos\theta(a))^2 + (v(a)-\cos\theta(a))^2$

$$= (u(a) - \cos\theta_0)^2 + (v(a) - \sin\theta_0)^2$$
$$= 0$$

$\therefore \quad u = \cos\theta, \ v = \sin\theta, \ \forall\, t \in [a,b].$ ※

Winding Number

Def: Let $z_0 \in \mathbb{C}$ and C be a closed contour (not necessary simple) such that $z_0 \notin C$. Then

$$\boxed{n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}} \quad \text{is an integer}$$

and is called the winding number of C with respect to z_0 or the index of z_0 with respect to C.

Remarks: (i) Under the assumptions of the Argument Principle, we have $\boxed{\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = n(f(C), 0)}$

(by change of variables) and hence

$$\boxed{n(f(C), 0) = N_0(f) - N_p(f)}$$

(ii) If C is parametrized by $z(t)$, $a \le t \le b$ $(z(a) = z(b))$

then $z(t) - z_0 \neq 0, \ \forall t$

$\Rightarrow \exists$ differentiable $\theta(t)$, $a \leq t \leq b$ s.t.

$$z(t) - z_0 = |z(t) - z_0| e^{i\theta(t)}$$

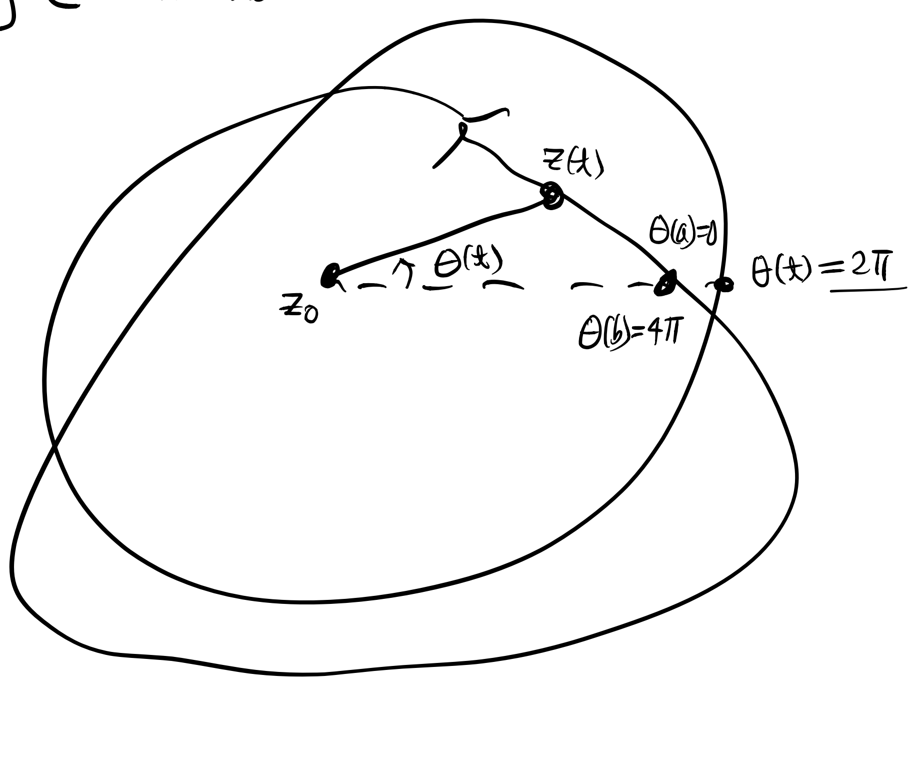
Hence
$$n(C, z_0) = \frac{1}{2\pi i} \int_a^b \frac{z'(t)\,dt}{z(t) - z_0}$$

$$= \frac{1}{2\pi i} \int_a^b \left[\frac{d}{dt} \ln|z(t) - z_0| + i\theta'(t)\right] dt$$

$$= \frac{1}{2\pi}\left[\theta(b) - \theta(a)\right]$$

$$= \frac{1}{2\pi} \Delta_C \arg(z(t) - z_0) \quad (\in \mathbb{Z})$$

can be interpreted as the number of turns made by following C around z_0.



eg1: $f(z) = z^n$, $C: |z| = 1$, +ve oriented $(n \in \mathbb{Z})$

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = n \quad \& \quad f(e^{it}) = e^{int}$$

$$0 \xrightarrow{t} 2\pi \Rightarrow 0 \xrightarrow{nt} 2n\pi$$

eg2: $f(z) = \dfrac{(z-8)^2 z^3}{(z-5)^4 (z+2)^2 (z-1)^5}$, $C: |z| = 4$, +ve oriented

zeros inside C: $z = 0$ order 3

poles inside C: $z = 1$ order 5; $z = -2$ order 2.

$$\Rightarrow \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = 3 - (5+2) = -4.$$

Rouché's Theorem

Thm 3 If f and g are analytic functions on and inside a simple closed contour C such that

$$|g(z)| < |f(z)|, \quad \forall z \in C$$

Then f and $f+g$ have the same number of zeros, counting multiplicities, inside C.

Pf: By assumption $|f(z)| > |g(z)| \geq 0$ and

$$|f(z)+g(z)| \geq |f(z)|-|g(z)| > 0 \quad \text{on } C$$

Hence $F(z) = \dfrac{f(z)+g(z)}{f(z)}$ is analytic and satisfies

$$|F(z)-1| = \left|\frac{g(z)}{f(z)}\right| < 1 \quad \text{on } C.$$

$\therefore$ 0 is not enclosed by the contour $F(C)$.

i.e. the contour $F(C)$ never "around" 0.

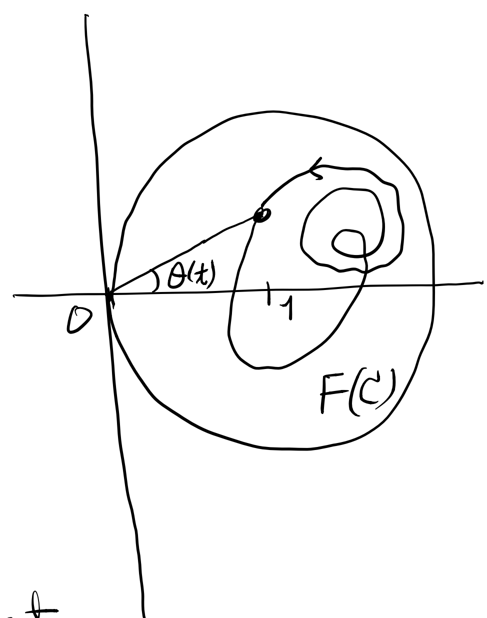
$\Rightarrow 0 = n(F(C), 0)$.

To see this more presicely, we note that

the smooth argument function $\theta(t)$ can be chosen so that

$$-\frac{\pi}{2} < \theta(t) < \frac{\pi}{2}, \quad t \in [a,b]$$

$$\therefore \quad \frac{1}{2\pi}|\theta(b)-\theta(a)| < \frac{1}{2}$$



But it is an integer, we must have $n(F(C),0) = \frac{1}{2\pi}(\theta(b)-\theta(a)) = 0$.

Hence

$$0 = \frac{1}{2\pi i}\int_C \frac{F'}{F}\,dz = \frac{1}{2\pi i}\int_C \frac{fg'-gf'}{f+g}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(g'+f')-(f+g)f'}{f(f+g)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(f+g)'}{f+g}\,dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$$

$$= N_0(f+g) - N_0(f) \quad \#$$

<u>eg3</u>: All the zeros of $h(z) = z^5 + z + 3$ lie inside $|z| < 2$.

<u>Soln</u>: Write $f(z) = z^5$ & $g(z) = z+3$

Then f, g analytic on and inside $|z| = 2$.

And $|g(z)| = |z+3| \le |z| + 3 = 5 < 2^5 = |f(z)|$

for z with $|z| = 2$.

By Rouché's Thm, $f(z) = z^5$ and $h(z) = f(z) + g(z)$ have the same number of zeros inside $|z| = 2$.

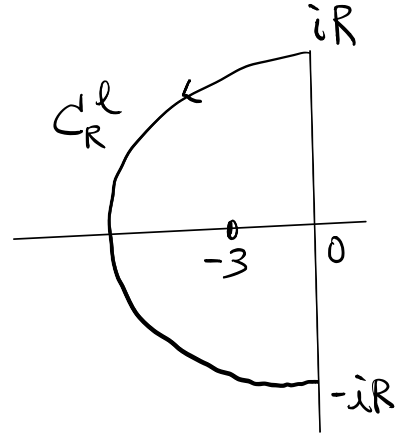
$\therefore$ $h(z)$ has 5 zeros inside $|z|=2$.

i.e. all zeros of $h(z)$ lie inside $|z|<2$.

<u>eg 4</u>: The equation $z+3+2e^z=0$ has precisely one root in the left half-plane.

<u>Soln</u>: Let $f(z) = z+3$

$g(z) = 2e^z$



For $R>0$ sufficiently large (says $R>6$),

$$|f(z)| \geq 3 \quad \text{on } C$$

and $$|g(z)| = |2e^z| = 2|e^{x+iy}| = 2e^x \leq 2e^0 \leq 2$$

$\therefore$ $|g(z)| < |f(z)|$ on C.

Hence $f+g = z+3+2e^z$ has the same number of zeros inside C as $f(z) = z+3$

$\therefore$ $z+3+2e^z$ has precisely one root inside C

Letting $R \to +\infty$, we see that

$z+3+2e^z$ has exactly one root in the left half-plane. ※

<u>Thm4</u> If $f(z)$ is analytic in a domain D and $z_0 \in D$ is a zero of $f(z)-w_0$ of order $m \geq 1$. Then $\exists\, \varepsilon>0,\ \delta>0$ such that

$\forall\, w \in \{0<|w-w_0|<\varepsilon\}$,

$f(z)-w$ has exactly m <u>distinct</u> zeros in $\{0<|z-z_0| \leq \delta\}$.

<u>Pf</u>: By assumption, we can find $\delta>0$ such that

- $\{|z-z_0| \leq \delta\} \subset D$ and
- $f(z)-w_0 \neq 0 \quad \forall\ 0<|z-z_0| \leq \delta$

As f' is also analytic, we can choose a smaller δ so that we also have

- $f'(z) \neq 0, \quad \forall\ 0<|z-z_0| \leq \delta$.

By compactness of $|z-z_0|=\delta$, $\exists\, \varepsilon>0$ such that

$$|f(z)-w_0| \geq \varepsilon, \quad \forall\ |z-z_0|=\delta.$$

Now for any w satisfies $0<|w-w_0|<\varepsilon$, we define $g(z)=f(z)-w$

Then $g(z)$ is analytic inside and on $|z-z_0|=\delta$.

Write $g(z) = (f(z)-w_0) + (w_0-w)$

$= F(z) \quad + \quad G(z)$ ← constant function.

Note that

$$|G(z)| = |w_0-w| < \varepsilon \leq |f(z)-w_0| = |F(z)|$$
$$\forall\, |z-z_0|=\delta.$$

$\therefore$ Rouché's Thm $\Rightarrow$ $g(z) = F(z)+G(z)$ has the same number of zeros insides $|z-z_0|=\delta$ as $F(z) = f(z)-w_0$.

Hence $g(z)$ has m zeros inside $|z-z_0|=\delta$ (as z_0 is the only zero of $F(z)$ (of order m) inside $|z-z_0|=\delta$)

Since $f'(z) \neq 0$ on $0<|z-z_0|\leq\delta$, all the zeros has order 1. Hence $g(z) = f(z)-w$ has exactly m distinct zeros in $0<|z-z_0|\leq\delta$. ##

<u>Cor 2</u>: If f is analytic at z_0 and $f'(z_0)\neq 0$, then f is one-to-one in a nbd. of z_0.

Pf: By assumption z_0 is a zero of order 1 of $f(z)-f(z_0) \Rightarrow \exists$ nbd of $w_0 = f(z_0)$ s.t. $\forall\, w$ in the nbd, $\exists$ exactly 1 zero of $f(z)-w$ in a nbd. of z_0

$\therefore$ f is 1-1 in this nbd. of z_0 $\times$

Open Mapping Theorem

Thm5: If f is analytic and non-constant in a domain D (connected open set), then f is open.

i.e. $f(D) = \{ w : w = f(z) \text{ for some } z \in D \}$ is an open set in $\mathbb{C}$.

i.e. $\forall\, w_0 = f(z_0) \in f(D)$, $\exists\, \varepsilon > 0$ such that

$$\{ |w - w_0| < \varepsilon \} \subset f(D).$$

Pf: Let $w_0 = f(z_0) \in f(D)$

Since f is nonconstant, z_0 is an isolated zero of $f(z) - w_0$. Hence, by Thm4, we can find $\varepsilon > 0$ and $\delta > 0$ such that $\forall\, w \in \{ 0 < |w - w_0| < \varepsilon \}$,

$f(z) - w$ has at least one zero in $0 < |z - z_0| \leq \delta$.

$\therefore \{ |w - w_0| < \varepsilon \} \subset f(D)$ (as $w_0 \in f(D)$).

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Remarks: (i) Rouché's Thm can be used to provide another proof of Fundamental Thm of Algebra (Ex!)

(ii) Open mapping thm can be used to prove the maximum modulus principle (Ex!)

Thm 6 (Hurwitz Theorem)

Let $\{f_n(z)\}$ be a sequence of analytic functions on a domain D. Suppose that there exists an analytic function $f(z)$, not identically zero, on D such that $\{f_n(z)\}$ converges to $f(z)$ uniformly on any compact subset $K \subset D$. Then for any simple closed contour C in D, not passing through zeros of $f(z)$, $\exists\, N > 0$ such that $f_n(z)$ and $f(z)$ has the same number of zeros, counting multiplicities, inside C, for all $n \geq N$.

Remark: In fact, we don't need to assume the analyticity of $f(z)$ as we'll prove later that f_n analytic & $f_n \to f$ uniformly in cpt subset implies f is analytic (Weierstrass Thm).

Pf of Hurwitz Thm

A simple closed contour C is always cpt.

Hence $f(z) \neq 0$, $\forall z \in C$, implies $\min_{z \in C} |f(z)| = \alpha > 0$.

By assumption, $f_n \to f$ uniformly on C.

$\therefore \exists N > 0$ such that $\forall n \geq N$, we have

$$|f_n(z) - f(z)| < \alpha, \quad \forall z \in C.$$

$$\Rightarrow \forall n \geq N, \quad |f_n(z) - f(z)| < |f(z)|, \quad \forall z \in C.$$

Rouché's Thm $\Rightarrow$ $f_n(z)$ & $f(z)$ have the same number of zero inside C ($\forall n \geq N$).

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