HW 6 Due Mar 30, 2017

1. Find all the isolated singular points, write down the principal part, classify the type, and compute the residue.

(a) $\dfrac{z-1}{z^2-5z+4}$, (b) $\sin\left(\dfrac{2}{z}\right)$, (c) $\dfrac{z+1}{\cos z}$

(d) $\dfrac{\sin 3z}{z}$, (e) $\dfrac{z^2}{2-\sqrt{z}}$ where $\sqrt{z}$ denotes the principal branch.

2. Using one residue to evaluate the following contour integrals, where $|z|=3$ is positively oriented.

(a) $\displaystyle\int_{|z|=3} \frac{2z-3}{z(z+1)}\,dz$, (b) $\displaystyle\int_{|z|=3} \frac{z^3}{4+z^2}\,dz$.

3. Suppose that $g(z)$ is analytic and has a zero of order 1 at z_0. Show that $f(z)=\dfrac{1}{[g(z)]^2}$ has a pole of order 2 at z_0 with residue

$$\operatorname*{Res}_{z=z_0} f(z) = -\frac{g''(z_0)}{[g'(z_0)]^3}.$$

4. For any $N>0$, let C_N be the boundary of the square bounded by the straight lines $x=\pm(N+\frac{1}{2})\pi$ and $y=\pm(N+\frac{1}{2})\pi$ in positive orientation.

(a) Show that
$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right].$$

(b) Using (a), show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$
by estimating $\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right|$ in terms of N.