§ 4.3 General Theory of (complex) Power Series

<u>Def</u>: The <u>greatest radius</u> $R>0$ such that

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ converges for } |z-z_0|<R$$

is called the <u>radius of convergence</u> and $|z-z_0|=R$ is called the <u>circle of convergence</u> of the series.

($R=0$ is not interesting for our discussion)

<u>Note</u>: R could be $+\infty$.

<u>Lemma 1</u> If $\sum_{n=0}^{\infty} a_n (z_1-z_0)^n$ converges, $(z_1 \neq z_0)$ then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is <u>absolutely convergent</u> for $|z-z_0| < |z_1-z_0|$

<u>Pf</u>: $\sum_{n=0}^{\infty} a_n (z_1-z_0)^n$ converges $\Rightarrow$ $a_n(z_1-z_0)^n \to 0$ as $n\to\infty$.

$$\Rightarrow \exists M>0 \text{ s.t. } |a_n(z_1-z_0)^n| \le M, \ \forall n=0,1,2,\cdots$$

Hence $\forall\ |z-z_0|<|z_1-z_0|$, we have

$$\sum_{n=0}^{\infty} |a_n(z-z_0)^n| = \sum_{n=0}^{\infty} |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n$$

$$\le M \sum_{n=0}^{\infty} \rho^n$$

where $\rho = \left|\dfrac{z-z_0}{z_1-z_0}\right| < 1$.

Hence $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges by comparison test $\#$

<u>Thm 1</u> Let $R>0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Then (1) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ absolutely convergent $\forall |z-z_0|<R$

(2) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges $\forall |z-z_0|>R$

<u>Pf</u>: (1) Let $|z-z_0|<R$, $\exists z_1$ s.t. $|z-z_0|<|z_1-z_0|<R$.

By definition of R,

$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges

$\therefore$ Lemma 1 $\Rightarrow \sum_{n=0}^{\infty} a_n(z-z_0)^n$ absolutely convergent.

(2) Suppose not, $\exists z_2$ s.t. $|z_2-z_0|>R$ s.t.

$\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ converges.

Then $\forall |z-z_0| < \dfrac{R+|z_2-z_0|}{2} < |z_2-z_0|$,

we have, by Lemma 1, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges.

Since $\dfrac{R+|z_2-z_0|}{2} > R$, this is a contradiction to the definition of R. $\#$

Thm 2 Let $R>0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Then

(1) $\forall\, 0<R_1<R$, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniformly convergent

in $|z-z_0|\le R_1$.

(2) Hence $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ defines a

continuous function on $|z-z_0|<R$.

Recall: Uniform convergent means

$\forall\, \varepsilon>0$, $\exists\, N_\varepsilon>0$ (indep. of z in $|z-z_0|\le R_1$)

such that

$$\left|\sum_{n=N}^{\infty} a_n(z-z_0)^n\right|<\varepsilon, \quad \begin{array}{l}\forall\, N>N_\varepsilon \\ \forall\, |z-z_0|\le R_1.\end{array}$$

Pf of (1) Pick any z_1 s.t. $|z_1-z_0|=R_1<R$.

Thm 1 $\Rightarrow$ $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ absolutely convergent

i.e. $\sum_{n=0}^{\infty} |a_n||z_1-z_0|^n$ converges.

$\Rightarrow$ $\forall\, \varepsilon>0$, $\exists\, N_\varepsilon>0$ such that

$$\sum_{n=N}^{\infty} |a_n||z_1-z_0|^n<\varepsilon, \quad \forall\, N>N_\varepsilon$$

i.e. $\displaystyle\sum_{n=N}^{\infty} |a_n| R_1^n < \varepsilon, \quad \forall N > N_\varepsilon.$

Hence $\forall N > N_\varepsilon$ & $|z - z_0| \le R_1$, we have

$$\left| \sum_{n=N}^{\infty} a_n (z-z_0)^n \right| \le \sum_{n=N}^{\infty} |a_n| \, |z-z_0|^n \le \sum_{n=N}^{\infty} |a_n| R_1^n$$

$$< \varepsilon. \quad \#$$

<u>Pf of (2)</u> $\forall\ |z^* - z_0| < R$, $\exists\ 0 < R_1 < R$ s.t.

$$|z^* - z_0| < R_1 < R.$$

Using (1), $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is uniform converges on $|z-z_0| \le R_1$

$\Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon > 0$ s.t.

$$(*)_1 \quad \left| \sum_{n=N}^{\infty} a_n (z-z_0)^n \right| < \varepsilon/3, \quad \forall N > N_\varepsilon, \ \forall |z-z_0| \le R_1$$

On the other hand, $\displaystyle\sum_{n=0}^{N_\varepsilon} a_n (z-z_0)^n$ is a polynomial and hence continuous on $|z-z_0| \le R_1$.

$\Rightarrow$ (for the same $\varepsilon > 0$) $\exists\ \delta > 0$ such that

$$(*)_2 \quad \left| \sum_{n=0}^{N_\varepsilon} a_n (z-z_0)^n - \sum_{n=0}^{N_\varepsilon} a_n (z^*-z_0)^n \right| < \varepsilon/3, \quad \forall |z - z^*| < \delta$$

and $|z - z_0| \le R_1$

Therefore, $\forall\, |z - z^*| < \delta$

$$\begin{aligned}
|S(z) - S(z^*)| &\le \left| S(z) - \sum_{n=0}^{N_\varepsilon} a_n (z-z_0)^n \right| \\
&\quad + \left| \sum_{n=0}^{N_\varepsilon} a_n (z-z_0)^n - \sum_{n=0}^{N_\varepsilon} a_n (z^*-z_0)^n \right| \\
&\quad + \left| S(z^*) - \sum_{n=0}^{N_\varepsilon} a_n (z^*-z_0)^n \right| \\
&< \underbrace{\frac{\varepsilon}{3}}_{\text{by } (*)_1} + \overbrace{\frac{\varepsilon}{3}}^{\text{by } (*)_2} + \underbrace{\frac{\varepsilon}{3}}_{\text{by } (*)_1} \\
&= \varepsilon
\end{aligned}$$

$\therefore$ $S(z)$ is continuous at z^*. Since z^* is arbitrary, we see that $S(z)$ is continuous on $|z - z_0| < R$. ⋕

<u>Thm 3</u> Let $R > 0$ be radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Then

(1) (Integration term-by-term)

$\forall$ contour C in $|z - z_0| < R$ and any continuous function g on C,

$$\int_C g(z) \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right) dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z-z_0)^n \, dz$$

(i.e. RHS is convergent and the sum equals LHS)

(2) $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is <u>analytic</u> in $|z-z_0|<R$.

(3) $S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ in $|z-z_0|<R$.

(differentiation term-by-term)

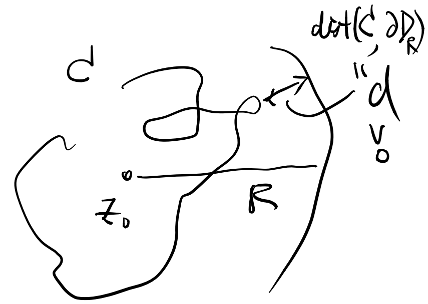
<u>Pf of (1)</u> Let $M = \max_{z\in C} |g(z)|$ and $L =$ length of C.

$$\left| \int_C g(z) \sum_{n=0}^{\infty} a_n (z-z_0)^n dz - \sum_{n=0}^{N} a_n \int_C g(z)(z-z_0)^n dz \right|$$

$$= \left| \int_C g(z) \left[\sum_{n=N+1}^{\infty} a_n (z-z_0)^n \right] dz \right| .$$

Since C is compact in $|z-z_0|<R$, $\exists\ 0<R_1<R$ such that C is contained in $|z-z_0| \le R_1 < R$.

Uniform convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ on $|z-z_0| \le R_1$ $\Rightarrow$



$\forall \varepsilon > 0$, $\exists N_\varepsilon > 0$ s.t.

$$\left| \sum_{n=N+1}^{\infty} a_n (z-z_0)^n \right| < \varepsilon, \quad \forall N+1 > N_\varepsilon$$
$$\forall |z-z_0| \le R_1 .$$

Hence $\left| \int_C g(z)\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)dz - \sum_{n=0}^{N} a_n \int_C g(z)(z-z_0)^n dz \right|$

$\leq M L \varepsilon.$

Hence $\int_C g(z)\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$

✗

Pf of (2)

Applying $g \equiv 1$ and any closed contour C in (1).

$\Rightarrow \quad \int_C S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz$

$= 0 \quad$ as $\int_C (z-z_0)^n dz = 0, \ \forall n = 0, 1, \cdots$

Then by Thm 2 of §3.9 (usually referred as Morera Thm),

$S(z)$ is analytic in $|z-z_0| < R$.

Pf of (3)

$\forall |z-z_0| < R$, $\exists R_1 > 0$ s.t. $|z-z_0| < R_1 < R$.

Then applying statement (1) to

$C : |s-z_0| = R_1$ positive oriented (counterclockwise)

and $g(s) = \frac{1}{2\pi i} \frac{1}{(s-z)^2}$ for $s \in C$, we have

$$\int_{|s-z_0|=R_1} \frac{S(s)}{2\pi i (s-z)^2} ds = \sum_{n=0}^{\infty} a_n \int_{|s-z_0|=R_1} \frac{(s-z_0)^n}{2\pi i (s-z)^2} ds$$

By statement (2) and Cauchy Integral Formula,

$$S'(z) = \frac{1}{2\pi i} \int_{|s-z_0|=R_1} \frac{S(s)}{(s-z)^2} ds \quad \left(\text{as } |z-z_0|<R_1\right)$$

Similarly by Cauchy Integral Formula, $\forall n = 0, 1, 2, \cdots$

$$\frac{1}{2\pi i} \int_{|s-z_0|=R_1} \frac{(s-z_0)^n}{(s-z)^2} ds = \frac{d}{ds}\Big|_{s=z} (s-z_0)^n$$

$$= n(z-z_0)^{n-1},$$

Hence

$$S'(z) = \sum_{n=0}^{\infty} a_n \cdot n (z-z_0)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

$\divideontimes$

<u>Cor 1</u>: Let $R>0$ be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. Then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the Taylor series of $f(z)$ at z_0,

i.e. $$a_n = \frac{1}{n!} f^{(n)}(z_0) \quad (n = 0, 1, 2, \cdots).$$

Pf: By (3) of Thm 3,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

$$\Rightarrow \quad f'(z_0) = a_1 .$$

By induction, it is easy to see that

$\forall\, k = 0, 1, 2, \cdots,$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-z_0)^{n-k} .$$

Hence $f^{(k)}(z_0) = k(k-1)\cdots 1 \cdot a_k = k!\, a_k$ ##

(We've used $f^{(k)}(z_0) = \dfrac{k!}{2\pi i} \displaystyle\int_C \frac{f(s)}{(s-z_0)^{k+1}}\, ds$, $\forall\, C$ enclosing z_0)

Remarks: (i) All the above can be extended (with modification) to Laurent series:

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

for both series on the RHS.

(ii) Multiplication, division & composition can be performed on complex power series as in the real case.

eg1: $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

$$\Rightarrow \frac{d}{dz}\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, z^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \cos z \quad .$$

eg2: $f(z) = \frac{\sinh z}{1+z}, \quad |z|<1$

$$= (\sinh z)\left(\frac{1}{1+z}\right)$$

$$= \left(z + \frac{z^3}{3!} + \cdots\right)\left(1 - z + z^2 - \cdots\right)$$

$$= z - z^2 + z^3 - z^4 + \cdots$$
$$+ \frac{1}{3!}z^3 - \frac{1}{3!}z^4 + \cdots \quad (\text{up to } z^4)$$

$$= z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \cdots \quad (|z|<1)$$

eg3 $\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$

$$= \frac{1}{z\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$= \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!}\left(1 + \frac{3!}{5!} z^2 + \cdots\right)}$$

$$= \frac{1}{z}\left[1 - \frac{z^2}{3!}\left(1 + \frac{3!}{5!} z^2 + \cdots\right) + \frac{z^4}{(3!)^2}\left(1 + \frac{3!}{5!} z^2 + \cdots\right)^2 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{z^2}{3!} - \frac{3!}{5!} z^4 + \cdots + \frac{z^4}{(3!)^2} + \cdots\right]$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{120} z^3 + \cdots \qquad (0 < |z| < \pi)$$

§4.4 Zeros and Uniqueness of Analytic Functions

Def: Suppose f is analytic at z_0. If there is a positive integer $m \geq 1$ such that

$$f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) \quad \text{and}$$

$$f^{(m)}(z_0) \neq 0,$$

then f is said to have a zero of order m at z_0.

Thm 1: Let f be analytic in $|z-z_0|<R$. Then f has a zero of order m at z_0 $\Longleftrightarrow$ there is an analytic function $g(z)$ such that

$$f(z) = (z-z_0)^m g(z) \quad \text{for } |z-z_0|<R,$$

and $\quad g(z_0) \neq 0.$

Pf: ($\Rightarrow$) Taylor's expansion (together with Cauchy Integral Formula)

$$f(z) = \cancel{f^{(0)}(z_0)}^{\,0} + \frac{\cancel{f^{(1)}(z_0)}^{\,0}}{1!}(z-z_0) + \dots + \frac{\cancel{f^{(m-1)}(z_0)}^{\,0}}{(m-1)!}(z-z_0)^{m-1}$$

$$+ \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \dots$$

$$= (z-z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots \right]$$

$\therefore \quad f(z) = (z-z_0)^m g(z)$,

where $\quad g(z) = \dfrac{f^{(m)}(z_0)}{m!} + \dfrac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots$

is analytic and

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0 .$$

($\Leftarrow$) If g analytic, then

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \cdots$$

$$\Rightarrow \quad f(z) = (z-z_0)^m \left[g(z_0) + g'(z_0)(z-z_0) + \cdots \right]$$

$$= g(z_0)(z-z_0)^m + g'(z_0)(z-z_0)^{m+1} + \cdots$$

By uniqueness of power series expansion, this is the Taylor series of $f(z)$ at z_0,

$$\therefore \quad f(z_0) = f^{(1)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

and $\quad f^{(m)}(z_0) = m!\, g(z_0) \neq 0$ ※

Thm 2 Suppose f is analytic in a domain D and z_0 is a zero of f, but $f(z) \not\equiv 0$ in D. Then $\exists\, \varepsilon > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$.

(i.e. only z_0 is a zero of f in $|z - z_0| < \varepsilon$, in other words, zeros of f are isolated!)

Pf: Step 1 If $\exists\, \varepsilon_1 > 0$ s.t. $f \not\equiv 0$ in $|z - z_0| < \varepsilon_1$

Then $\exists\ 0 < \varepsilon_2 < \varepsilon_1$ such that

$$f(z) \neq 0 \text{ in } 0 < |z - z_0| < \varepsilon_2 .$$

Pf of Step 1 :

$f \not\equiv 0$ in $|z - z_0| < \varepsilon_1$

$\Rightarrow$ Taylor's expansion of f at $z_0 \not\equiv 0$

$\Rightarrow$ z_0 is a zero of order m for some finite $m \geq 1$.

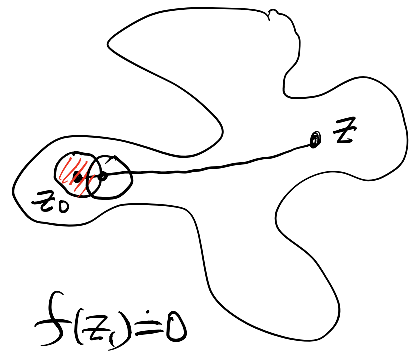
$\overset{(\text{Thm} 1)}{\Rightarrow}$ $f(z) = (z - z_0)^m g(z)$ with g analytic in $|z - z_0| < \varepsilon_1$

and $g(z_0) \neq 0$.

Continuity $\Rightarrow$ $\exists\, \varepsilon_2 > 0$ $(< \varepsilon_1)$ s.t. $g(z) \neq 0$ $\forall$ $|z - z_0| < \varepsilon_2 < \varepsilon_1$

$\Rightarrow$ $f(z) = (z - z_0)^m g(z) \neq 0$ $\forall$ $0 < |z - z_0| < \varepsilon_2$

This proves Step 1.



<u>Step 2</u> : General case.

<u>Pf of Step 2</u> : Suppose not,

$\forall\, \varepsilon_1 > 0$, there exists z_1

in $0 < |z - z_0| < \varepsilon_1$ such that $f(z_1) = 0$

Applying Step 1, $f(z) \equiv 0$ in $|z - z_0| < \varepsilon_1$.

Now $\forall\, z \in D \setminus \{z_0\}$. Connect z to z_0 by a path C.

Let ζ_1 be the last intersection point of $C \cap |z - z_0| = \varepsilon_1$.

Then continuity of $f \Rightarrow f(\zeta_1) = 0$.

Since $f \equiv 0$ on $C \cap \{|z - z_0| < \varepsilon_1\}$, ζ_1 is not isolated.

By Step 1 again, $\exists\, \varepsilon_2 > 0$ such that

$$f(z) \equiv 0 \text{ in } |z - \zeta_1| < \varepsilon_2 .$$

Repeating this process and using compactness of C,

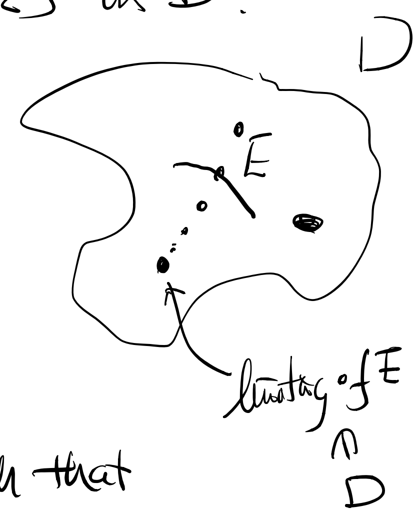
we see that $f(z) = 0$. Since $z \in D \setminus \{z_0\}$ is

arbitrary, we have shown that if z_0 is not isolated,

then $f(z) \equiv 0 \;\; \forall\, z \in D$. $\#$

Thm 3 (Uniqueness of analytic function)

Suppose that f and g are analytic in a domain D such that $f(z) = g(z)$, $\forall z \in E$, where $E \subset D$ is a subset of D that contains a limiting point which is also in D. Then $f(z) \equiv g(z)$ in D.



(If $\exists \{z_n\}$ in D s.t. $z_n \to z^* \in D$,
$f(z_n) = g(z_n)$, $\forall n$,
then $f(z) \equiv g(z)$ in D)

Pf: By assumption, $\exists\, z^* \in D$ such that
z^* is an non-isolated zero of $(f-g)(z)$.
(By continuity of f & g).
Hence Thm 2 $\Rightarrow$ $(f-g)(z) \equiv 0$ in D. #