3.11 The Logarithmic Function

Def: The (multiple-valued) logarithmic function of

$z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$ is

$$\log z \stackrel{\text{def}}{=\!=} \ln r + i \arg z$$
$$= \ln r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \cdots$$

where $\ln r$ is the natural log on $\mathbb{R}^+$.

Notes: (1) $\log z$ is a set

$$\underbrace{\log z}_{\text{sets}} = \ln |z| + i \underbrace{\arg z}_{\text{sets}}$$

(2) $\forall\, w \in \log z$, we have

$$e^w = e^{\ln|z| + i\theta} \qquad \text{where } \theta \in \arg z.$$
$$= e^{\ln|z|} e^{i\theta}$$
$$= |z| e^{i\theta} = z.$$

$\therefore$ $\log z$ is the "inverse" of exponential function !

(3) However $\log(e^z) = \log(e^{x+iy}) = \ln e^x + i(y + 2k\pi)$

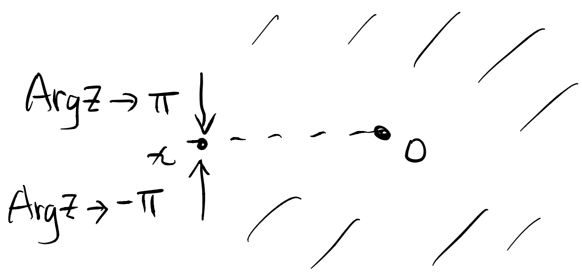
$(k = 0, \pm 1, \pm 2, \cdots)$ $\qquad = (x + iy) + 2k\pi i = z + 2k\pi i.$

Def: Principal value of $\log z$, denoted by $\mathrm{Log}\, z$, is

$$\boxed{\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z} \in \mathbb{C}$$

where $\mathrm{Arg}\, z$ = principal argument of $z \in (-\pi, \pi]$

Note: $\mathrm{Log}\, z$ is discontinuous at $z = x \le 0$.



Hence non-differentiable on $\mathbb{C}\setminus\{0\}$,

$\therefore$ non-analytic on $\mathbb{C}\setminus\{0\}$.

Derivatives of Log

If "$\log z$" is analytic near a point, by

$$\log z = \ln r + i\theta$$

we have $u = \ln r$, $v = \theta$

$$\Rightarrow \begin{cases} u_r = \frac{1}{r} = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = 0 = -v_r \end{cases} \quad \text{(C-R eqt.)}$$

$u_r, u_\theta, v_r, v_\theta$ continuous & satisfy CR eqt.

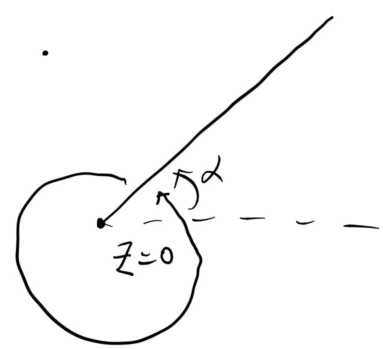
$$\Rightarrow \quad \frac{d}{dz}\log z = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

$\therefore$ If such a $\log z$ is defined and analytic near a point z, then

$$\boxed{\frac{d}{dz}\log z = \frac{1}{z}.}$$

Note that $\int_C \frac{dz}{z} = \pm 2\pi i \neq 0$ for simple closed contour enclosing $z=0$, we see that $\log z$ cannot be defined as an analytic function on the whole $\mathbb{C}\setminus\{0\}$, otherwise $\frac{1}{z}$ has a primitive $\log z$ in $\mathbb{C}\setminus\{0\} \Rightarrow \int_C \frac{dz}{z} = 0$.

- If we want to define an analytic $\log z$, then we need to aviod closed contours that enclosing $z=0$. So we define



<u>Branches of Log.</u>

<u>Def</u>: For any $\alpha \in \mathbb{R}$,

$$\boxed{\log z = \ln r + i\theta, \quad r>0, \ \alpha < \theta < \alpha + 2\pi}$$

$(z = re^{i\theta})$

is called a <u>branch of $\log z$</u>.

<u>Def</u>: The branch defined by $\alpha = -\pi$ is called the <u>Principal Branch</u> of log, denoted by

$$\boxed{\operatorname{Log} z = \ln r + i\Theta, \quad r>0, \ -\pi < \Theta < \pi.}$$

(strict inequality. different from Principal value!)

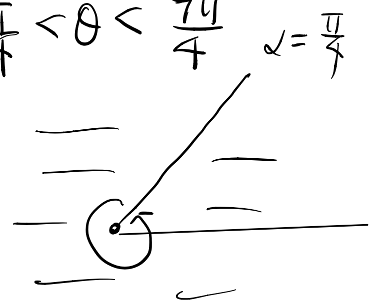
<u>Fact</u>: Same argument as in the calculation of the derivative of "$\log z$", we see that all branches of $\log z$ are analytic in its domains and satisfies

$$\boxed{\frac{d}{dz}\log z = \frac{1}{z}}$$

<u>Terminology</u>: The ray $\{\theta = \alpha\}$ is called the <u>branch cut</u> of the branch.

<u>eg</u> : (1) Branch of $\log z$ for $\alpha = \frac{\pi}{4}$

$\log z = \ln r + i\theta$, $\frac{\pi}{4} < \theta < \frac{9\pi}{4}$



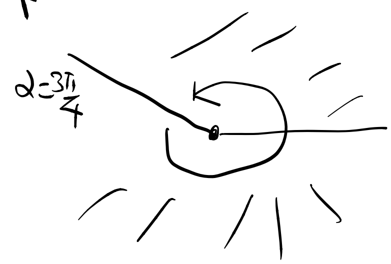
<u>Ex</u>: $\log i^2 = 2 \log i$

for this branch.

(2) Branch of $\log z$ for $\alpha = \frac{3\pi}{4}$

$\log z = \ln r + i\theta$,

$\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$.



<u>Ex</u>! $\log i^2 \neq 2 \log i$

for this branch!

But we still have

<u>Prop</u>: $\forall z_1, z_2 \in \mathbb{C} \setminus \{0\}$

$\log(z_1 z_2) = \log z_1 + \log z_2$ as <u>sets</u>!

(not branches)

(Pf: Ex!)

<u>Power Function</u>

<u>Def</u>: $\forall$ complex number c, we define the <u>power function</u>

by $z^c \stackrel{\text{def}}{=} e^{c \log z}$ (for $z \neq 0$)

Notes: (1) z^c is possibly multiple-valued.

(2) If $c = n \in \mathbb{Z}$, then

$$z^n = e^{n\log z} = e^{n[\ln r + i(\Theta + 2k\pi)]}$$

$$= e^{n\ln r + in\Theta}\, e^{i2(nk)\pi}$$

$$= e^{n\,\mathrm{Log}\, z} \quad \text{single-valued!}$$

(In fact, z^n is analytic)

(3) $c = \frac{1}{n}$, $n \in \mathbb{Z}\setminus\{0\}$

$$z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}[\ln r + i(\Theta + 2k\pi)]}, \quad k \in \mathbb{Z}$$

$$= \sqrt[n]{r}\, e^{i\left(\frac{\Theta}{n} + 2\left(\frac{k}{n}\right)\pi\right)} \qquad k = 0, 1, \cdots, n-1$$

$$= \text{set of } n\text{-roots of } z.$$

Def: A branch of z^c is the function defined on the domain of a branch of $\log z$ with value given by

$$z^c = e^{c\log z}, \quad r > 0, \ \alpha < \theta < \alpha + 2\pi,$$

with the corresponding branch of log.

Prop: For any branch of z^c,

$$\frac{d}{dz} z^c = c z^{c-1} \quad (r>0, \ \alpha < \theta < \alpha + 2\pi)$$

(Pf: Ex!)

Def: Principal value of z^c, denoted by

$$\text{P.V. } z^c \stackrel{\text{def}}{=} e^{c\,\text{Log} z}$$

where $\text{Log} z$ = principal value of log.

Exponential function with base c

$$c^z \stackrel{\text{def}}{=} e^{z \log c}$$

$$= e^{z[\ln|c| + i(\text{Arg} c + 2k\pi)]} \qquad k \in \mathbb{Z}$$

(is multiple-valued.)

Inverse Trigonometric & Hyperbolic Functions

(1) $w = \sin^{-1} z$

Soln: $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$$\Rightarrow \quad e^{iw} - 2iz - e^{-iw} = 0$$

$$\Rightarrow \quad (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$$

$$\Rightarrow \quad e^{iw} = \frac{2iz + [(2iz)^2 + 4]^{1/2}}{2} \leftarrow \text{multiple-valued}$$

$$= iz + (1 - z^2)^{1/2}$$

$$\Rightarrow \quad w = -i \log[iz + (1 - z^2)^{1/2}]$$

$$\boxed{\sin^{-1} z = -i \log\left[iz + (1-z^2)^{1/2}\right]}$$

multiple-valued function.

Similarly

(2) $$\boxed{\cos^{-1} z = -i \log\left[z + i(1-z^2)^{1/2}\right]}$$

(3) $$\boxed{\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}}$$

(4) $$\boxed{\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right]}$$

(5) $$\boxed{\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right]}$$

(6) $$\boxed{\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}}$$

(Ex: What are the domains of definition if principal branch of log is used in the formulae (3) and (6)?)

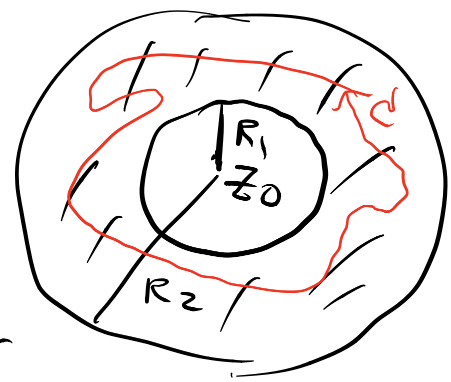
Ch4 Series

Thm 1 Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$

$(0 \leq R_1 < R_2 \leq +\infty)$

centered at z_0, let

C denote any positively oriented simple closed contour around z_0 and lying in the domain. Then at each point z in the domain, $f(z)$ has the series representation (Laurent Series)



$$(*)_1 \quad \boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad (R_1 < |z - z_0| < R_2)}$$

where

$$\boxed{c_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds, \quad \forall\, n = 0, \pm 1, \pm 2, \cdots}$$

Moreover, if f is actually analytic in $|z - z_0| < R_2$, then $c_n = 0$ for $n = -1, -2, -3, \cdots$ and we have

(Taylor Series)

$(*)_2$ $$\boxed{f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad |z - z_0| < R_2}$$

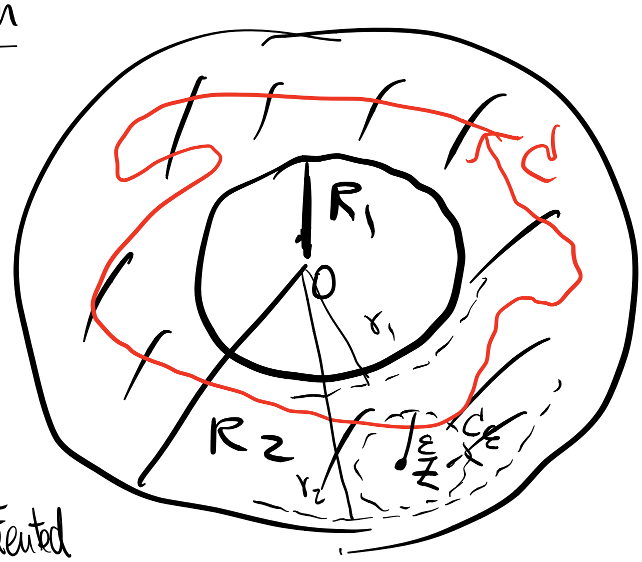
Note: $(*)_1$ means both infinite series

$$\sum_{n=0}^{+\infty} c_n (z - z_0)^n \quad \text{and}$$

$$\sum_{k=1}^{+\infty} c_{(-k)} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{c_{(-k)}}{(z - z_0)^k}$$

converge for all z in $R_1 < |z - z_0| < R_2$, and their sum equals $f(z)$.

Pf of Laurent's Theorem



Case $z_0 = 0$

Consider $\{ r_1 \le |z| \le r_2 \}$

with $R_1 < r_1 < r_2 < R_2$

Let $C_1 : |z| = r_1$

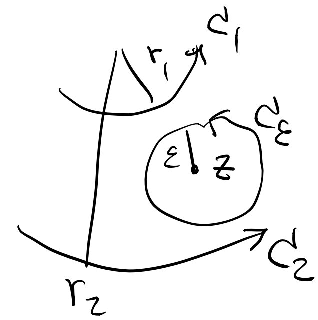
$C_2 : |z| = r_2$ +ve oriented

Then f is analytic on C_1 & C_2, and between them.

Let $z \in \{ r_1 < |z| < r_2 \}$. Then $\exists \varepsilon > 0$ s.t.

$D_\varepsilon(z) \subset \{r_1 < |z| < r_2\}$ with bdy C_ε (+ve oriented).

Applying Cauchy-Goursat Thm to the analytic function $\frac{f(s)}{s-z}$, we have



$$\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z} - \int_{C_\varepsilon} \frac{f(s)\,ds}{s-z} = 0$$

Then Cauchy Integral Formula $(n=0)$

$$f(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\left[\int_{C_2} \frac{f(s)\,ds}{s-z} - \int_{C_1} \frac{f(s)\,ds}{s-z}\right].$$

For $s \in C_2$, $\left|\frac{z}{s}\right| = \frac{|z|}{r_2} < 1$, we have

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s} \cdot \frac{1}{1-\frac{z}{s}}\,ds$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s}\left[\sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right] ds$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s^{n+1}}\,ds\right) z^n + \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds$$

Let $\rho_N(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$

and $M_2 = \sup_{\substack{s \in C_2 \\ (|s| = r_2)}} |f(s)|$

Then $|s-z| \geq \big| |s| - |z| \big| = r_2 - r$, where $r = |z| < r_2$.

$$\therefore |\rho_N(z)| \leq \frac{1}{2\pi} \cdot \frac{M_2}{r_2 - r} \left(\frac{r}{r_2}\right)^N \cdot 2\pi r_2$$

$$= \frac{M_2 r_2}{r_2 - r} \cdot \left(\frac{r}{r_2}\right)^N \to 0 \text{ as } N \to +\infty \quad \left(\text{since } \frac{r}{r_2} < 1\right)$$

$$\therefore \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds = \sum_{n=0}^{\infty} c_n z^n,$$

where $c_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s^{n+1}} ds$

by principle of deformation of paths.

For $s \in C_1$, $\left|\frac{z}{s}\right| = \frac{|z|}{r_1} = \frac{r}{r_1} > 1$.

Hence $-\frac{1}{2\pi i} \int_{C_1} \frac{f(s) ds}{s-z} = \frac{1}{2\pi i} \int_{C_1} \frac{f(s) ds}{z\left(1 - \frac{s}{z}\right)}$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z}\left[\sum_{n=0}^{N-1}\left(\frac{s}{z}\right)^n + \frac{\left(\frac{s}{z}\right)^N}{1-\frac{s}{z}}\right]ds$$

$$= \sum_{\substack{\cancel{n=0}\\ k=1}}^{\cancel{N-1}\ N} \left(\frac{1}{2\pi i}\int_{C_1} f(s)\, s^{\cancel{n}\,k-1}\, ds\right)\frac{1}{z^{\cancel{n+1}\,k}} \qquad (\text{let } k=n+1)$$

$$+ \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

$$= \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1} f(s)\, s^{n-1}\, ds\right)\frac{1}{z^n}$$

$$+ \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds$$

Now $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds\right|$

$$\le \frac{1}{2\pi}\,\frac{M_1}{r-r_1}\left(\frac{r_1}{r}\right)^N 2\pi r_1$$

$$= \frac{M_1 r_1}{r-r_1}\left(\frac{r_1}{r}\right)^N \to 0 \text{ as } N\to\infty \quad \left(\text{since } \frac{r_1}{r}<1\right)$$

where $M_1 = \sup_{\substack{s\in C_1\\ (|s|=r_1)}} |f(s)|$.

$$\therefore\ -\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s-z}\,ds = \sum_{n=1}^{\infty}\frac{c_{-n}}{z^n}$$

where $\quad c_{-n} = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s^{-n+1}} ds$

by principle of deformation of paths.

$\therefore \quad f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$

Since $r_1 < r_2$ satisfying $R_1 < r_1 < r_2 < R_2$ are arbitrary, we have proved $(*)_1$ for $z_0 = 0$.

General case $z_0 \neq 0$ follows easily by translation.

Finally, if f is actually analytic in $|z - z_0| < R_2$, then $\frac{f(s)}{s-z}$ is analytic inside and on C_1

$\Rightarrow \quad \int_{C_1} \frac{f(s)}{s-z} ds = 0 \Rightarrow \quad c_{-n} = 0, \ \forall n = 1, 2, 3, \cdots$

$\left(\text{or } c_{-n} = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds = \frac{1}{2\pi i} \int_{C_1} f(s) s^{n-1} ds = 0 \right.$

as $f(s) s^{n-1}$ analytic inside & on C_1. $\Big)$

#

egs: (1) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad (|z| < 1)$

(2) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \quad (|z| < \infty)$

(3) $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad (|z|<\infty)$

(4) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad (|z|<\infty)$

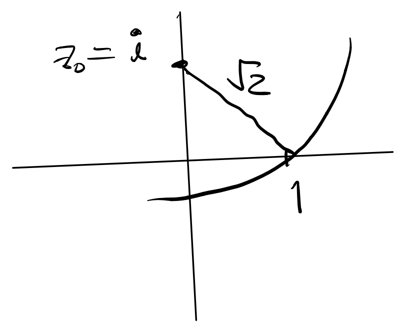
(5) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \quad (|z|<\infty)$

(6) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \quad (|z|<\infty)$

(7) $\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad (|z|<1)$

$\uparrow$ principal branch.

eg: $f(z) = \frac{1}{1-z}$ is analytic at $z_0 = i$



In fact, $f(z) = \frac{1}{1-z}$ is analytic in $|z-i|<\sqrt{2}$.

To find the Taylor series

$$f(z) = \frac{1}{1-z} = \frac{1}{(1-i)-(z-i)}$$

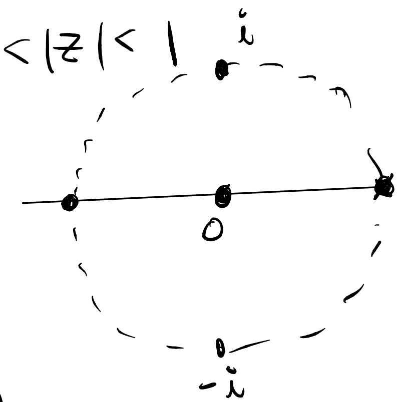
$$= \frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n \quad \text{as } \left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{\sqrt{2}} < 1.$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} \cdot (z-i)^n \quad \#$$

(We'll prove later that power series expansion is unique.)

<u>eg</u> $f(z) = \dfrac{1}{z(1+z^2)}$ on $0 < |z| < 1$



$$= \frac{1}{z} \, \frac{1}{1-(-z^2)}$$

$$= \frac{1}{z}\left(1 - z^2 + z^4 - z^6 + \cdots\right)$$

$$= \frac{1}{z} - z + z^3 - z^5 + \cdots \qquad (0 < |z| < 1)$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$