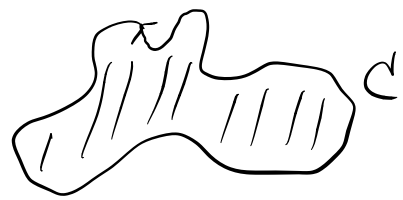
§3.5 Cauchy - Goursat Theorem

Thm (Cauchy - Goursat Theorem) If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)\,dz = 0$.



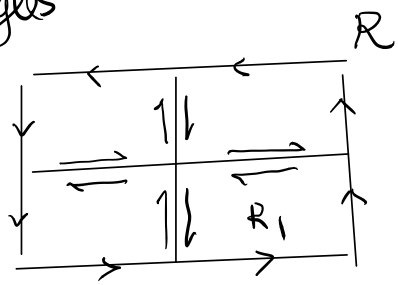
Easy version: (Cauchy Theorem) Same statement with additional condition that $f'(z)$ is continuous at all points interior to and on the simple closed contour C. (Just by Green's Thm.)

Lemma 1 (Cauchy-Goursat Theorem for Rectangles and triangles) If f is analytic on a closed rectangle (or triangle) R, then

$$\int_{\partial R} f(z)\,dz = 0 \quad (\text{where } \partial R = \text{boundary of } R)$$

Pf Let $\eta(R) = \int_{\partial R} f(z)\,dz$.

Divides R into 4 congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{4}$ by bisection



Then

$$\int_{\partial R} f(z)dz = \int_{\partial R^{(1)}} f(z)dz + \int_{\partial R^{(2)}} f(z)dz$$
$$+ \int_{\partial R^{(3)}} f(z)dz + \int_{\partial R^{(4)}} f(z)dz$$

since integrals over common sides (inside R) cancel each other. Hence

$$\eta(R) = \eta(R^{(1)}) + \cdots + \eta(R^{(4)})$$

$\Rightarrow$ $\exists$ a smallest k ($k=1,2,3,4$) such that

$$|\eta(R^{(k)})| \geq \frac{1}{4}|\eta(R)|$$

And denote this $R^{(k)}$ by R_1.

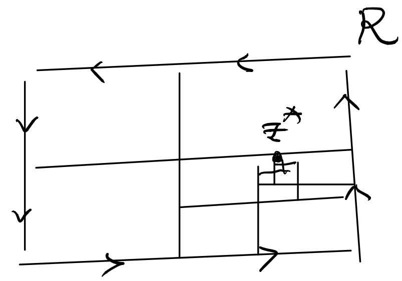
Repeating this process, we obtain a sequence of <u>nested</u> rectangles $R \supset R_1 \supset R_2 \cdots \supset R_n \supset \cdots$ such that

$$|\eta(R_n)| \geq \frac{1}{4}|\eta(R_{n-1})|, \quad \forall n=1,2,\cdots.$$

Hence $|\eta(R_n)| \geq \frac{1}{4}|\eta(R_{n-1})| \geq \frac{1}{4^2}|\eta(R_{n-2})| \geq \cdots \geq \frac{1}{4^n}|\eta(R)|$.

By completeness of $\mathbb{R}^2$, $\{R_n\}$ converges to a point z^* in R.

i.e. $\forall \delta > 0$, $\exists n_0 > 0$ such that



$$R_n \subset \{ z : |z - z^*| < \delta \},$$

$$\forall n \geq n_0 .$$

Now $\forall \varepsilon > 0$, $\exists \delta > 0$ such that f is analytic in $|z - z^*| < \delta$ and

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \varepsilon . \quad (0 < |z - z^*| < \delta)$$

Observe that $\frac{d}{dz} z = 1$ and $\frac{d}{dz}\left(\frac{1}{2} z^2\right) = z$ in the whole $\mathbb{C}$,

$$\int_{\partial R_n} dz = \int_{\partial R_n} z\, dz = 0 , \quad \forall n = 1, 2, \cdots$$

Then

$$\begin{aligned} \eta(R_n) &= \int_{\partial R_n} f(z)\, dz \\ &= \int_{\partial R_n} \left[f(z) - f(z^*) - f'(z^*)(z - z^*) \right] dz \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad |\eta(R_n)| &\leq \int_{\partial R_n} | f(z) - f(z^*) - f^*(z^*)(z - z^*) | \, dz \\ &\leq \varepsilon \int_{\partial R_n} |z - z^*| \, dz \end{aligned}$$

Note that d_n = diagonal of R_n and
L_n = perimeter of R_n

are given by $d_n = \frac{1}{2^n} d$ and $L_n = \frac{1}{2^n} L$

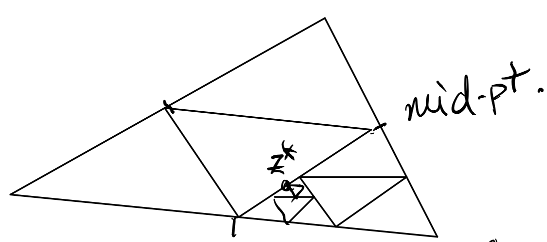
where d = diagonal of R and L = perimeter of R.

$$\therefore \quad |\eta(R_n)| \le \varepsilon \cdot \frac{d}{2^n} \cdot \frac{L}{2^n} = \frac{\varepsilon\, d \cdot L}{4^n}$$

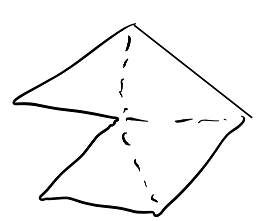
$$\therefore \quad |\eta(R)| \le 4^n |\eta(R_n)| \le \varepsilon \cdot d \cdot L.$$

Since $\varepsilon > 0$ is arbitrary, we have $\eta(R) = 0$.

For triangle, similar proof works :



Lemma 2 The Cauchy-Goursat Thm is true for <u>piecewise linear</u> simple closed contour (i.e. polygonal simple closed contour.) C.



<u>Pf</u>: We use induction on the

number of sides n ($n \geq 3$)

For $n=3$, C' is a triangle. Lemma 1 $\Rightarrow$ the result!

Assume the thm is true for polygonal simple closed contour with at most k sides.

For any polygonal simple closed contour with $k+1$ sides, we can divide it into two adjecent polygonal simple closed contours C'_1 & C'_2 with at most k sides, and one common (interior) line segment.

Induction hypothesis $\Rightarrow$

$$\int_{C'} f(z)\,dz = \int_{C'_1} f(z)\,dz + \int_{C'_2} f(z)\,dz \quad \left(\begin{array}{c}\text{cancellation} \\ \text{on the common} \\ \text{segment}\end{array}\right)$$

$$= 0 + 0.$$

This completes the proof. ※

Lemma 3 (Cauchy-Goursat Theorem for Disks (Convex domain))

If $f(z)$ is analytic in an open disk $D = \{|z-z_0| < r\}$ then $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in D.

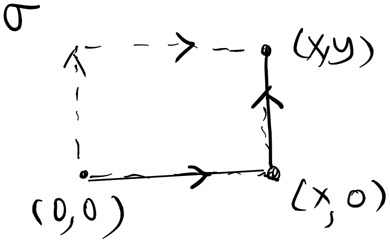
Pf: It's sufficient to prove that $f(z)$ has an antiderivation $F(z)$ in D.

After translation, we may assume $D = \{|z| < r\}$.

Define for $z = x+iy$ in D,

$$F(z) = \int_\sigma f(z)\,dz = \int_0^x f(t)\,dt + \int_0^y f(x+is)\,i\,ds$$

where σ :



Then by Cauchy-Goursat Thm for Rectangles, (Lemma 1)

we also have

$$F(z) = \int_0^y f(is)\,i\,ds + \int_0^x f(t+iy)\,dt$$

Then $$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{\left(\int_0^x f(t)dt + \int_0^{y+\Delta y} f(x+is)i\,ds\right) - \left(\int_0^x f(t)dt + \int_0^y f(x+is)i\,ds\right)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{i}{\Delta y} \int_y^{y+\Delta y} f(x+is)\,ds = i\,f(x+iy)$$

and $$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{\left(\int_0^y f(is)i\,ds + \int_0^{x+\Delta x} f(t+iy)\,dt\right) - \left(\int_0^y f(is)i\,ds + \int_0^x f(t+iy)dt\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t+iy)\,dt = f(x+iy)$$

i.e. $\frac{\partial}{\partial x}(\mathrm{Re}F + i\,\mathrm{Im}F) = f = \mathrm{Re}f + i\,\mathrm{Im}f$

$$= -i\frac{\partial}{\partial y}(\operatorname{Re}F + i\operatorname{Im}F)$$

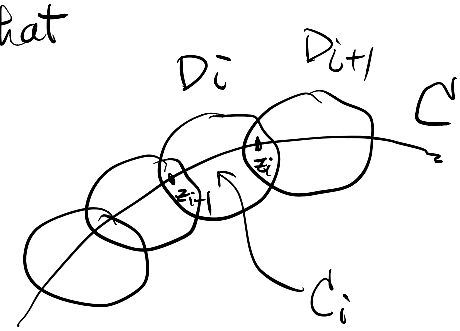
$$\Rightarrow \quad \frac{\partial \operatorname{Re}F}{\partial x} = \frac{\partial}{\partial y}\operatorname{Im}F, \quad \frac{\partial}{\partial x}\operatorname{Im}F = -\frac{\partial}{\partial y}\operatorname{Re}F$$

$\therefore$ F satisfies CR equations and

$$F'(z) = f(z). \quad \#$$

Proof of Cauchy-Goursat Theorem:

Since the simple closed contour C is compact and f is analytic on C, we can find finitely many open disks $D_1, D_2, \cdots, D_N$ such that



(i) $C \subset \bigcup_{i=1}^{N} D_i$

(ii) $\begin{cases} D_i \cap D_{i+1} \neq \phi, & i=1,\cdots,N-1 \\ D_N \cap D_1 \neq \phi \end{cases}$

(iii) $\exists$ points $z_1, \cdots, z_N$ on C such that

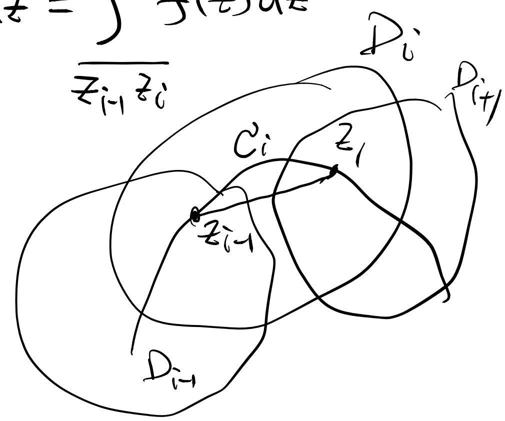
$$\begin{cases} z_i \in D_i \cap D_{i+1}, & i=1,\cdots,N-1 \\ z_N \in D_N \cap D_1 \end{cases}$$

such ~~that~~ the part of C between z_{i-1} to z_i denoted by C_i, contains completely in D_i,

and (iv) f analytic on D_i, $i=1,2,\cdots,N$

(⇒ f analytic on $\left(\bigcup_{i=1}^{N} D_i\right) \cup (\text{Interior of } C)$)

Therefore, Lemma 3 ⇒ $\int_{C_i} f(z)\,dz = \int_{\overline{z_{i-1} z_i}} f(z)\,dz$



(for $i = N$, z_{N+1} means z_1)

Since $\tilde{C} = \sum \left(\overline{z_i z_{i+1}}\right)$ is a piecewise linear contour, and f is analytic on and inside $\tilde{C}$, lemma 2 ⇒

$$\int_{\tilde{C}} f(z)\,dz = 0.$$

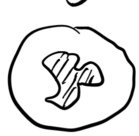
Hence $\int_C f(z)\,dz = \int_{\tilde{C}} f(z)\,dz = 0.$

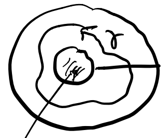
This completes the proof of the Cauchy-Goursat Theorem.

#

§3.6 Simply-Connected Domains

<u>Def</u>: A <u>simply connected domain</u> D is a domain such that every simple closed contour within it encloses only points of D.

<u>eg</u>: Disk  simply connected

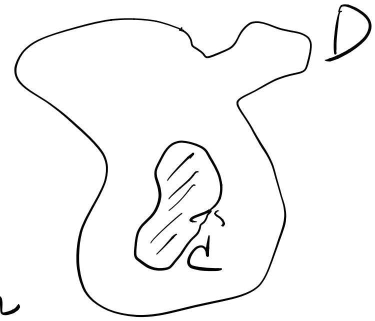
Annulus  $\{z : R_1 < |z - z_0| < R_2\}$ is <u>not</u> simply connected $(0 \le R_1 < R_2 \le +\infty)$

(γ encloses pts not in the annulus.)

<u>Thm</u> If a function f is <u>analytic</u> throughout a <u>simply connected domain</u> D, then

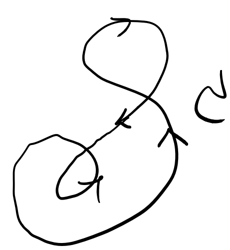
$$\int_C f(z)\,dz = 0$$

for <u>every closed contour</u> (not necessary simple) C lying in D.

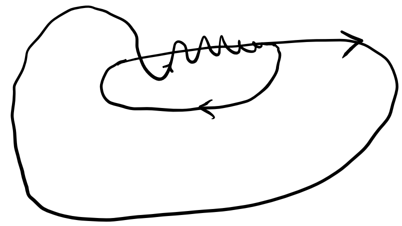


<u>Pf</u>: If C is simple, then the region enclosed by C is contained in D, $\therefore f$ is analytic interior to and on C. Cauchy-Goursat Thm $\Rightarrow \int_C f(z)dz = 0$.

If C is not simple, with finitely many self-intersections. Then C can be subdivided into finitely many simple closed contours. Then



$$\int_C f(z)\,dz = \sum_i \int_{C_i} f(z)\,dz = 0 \quad \text{as } \int_{C_i} f(z)\,dz = 0.$$



For infinitely many self-intersections, we replace C by a polygonal contour $\tilde{C}$ as in the proof of Cauchy-Goursat thm such that

$$\int_C f(z)\,dz = \int_{\tilde{C}} f(z)\,dz.$$

Since $\tilde{C}$ is polygonal, it only has finitely many (or none) self-intersections, $\int_{\tilde{C}} f(z)\,dz = 0$.

$$\therefore \int_C f(z)\,dz = 0. \quad \#$$

<u>Cor 1</u> : A function f that is <u>analytic</u> throughout a <u>simply-connected</u> domain D must have an antiderivative everywhere in D.

<u>Cor 2</u> : Entire functions always possess antiderivatives.
(Pf: $\mathbb{C}$ is simply-connected.)

§3.7 Multiple Connected Domains

Def: A domain that is not simply-connected is said to be multiple connected.

Thm: Suppose that

(a) C is a simple closed contour in counterclockwise direction;

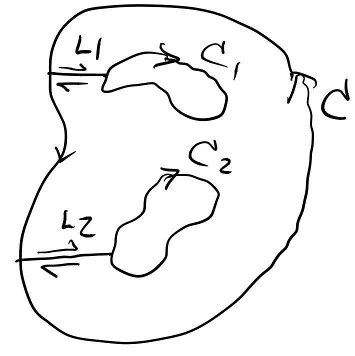
(b) C_k, $k=1,\cdots,n$ are simple closed contours interior to C in clockwise direction, they are disjoint and whose interiors are also disjoint.

If a function f is analytic on C and C_k, $k=1,\cdots,n$ and throughout the multiple connected domain consisting the points interior to C, but exterior to C_k, $k=1,2,\cdots,n$, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 .$$

Pf: Let L_k be contours joining C to C_k, $k=1,\cdots,n$ in the multiple connected domain such that L_k has no self-intersection, and L_k are disjoint.

Then a "simple closed" contour Γ can be formed:



$$\Gamma = \text{"}C\text{"} + L_1 + C_1 + (-L_1) + \cdots + L_n + C_n + (-L_n).$$

By Cauchy-Goursat Thm,

$$0 = \int_\Gamma f(z)\,dz = \left(\int_C + \int_{L_1} + \int_{C_1} + \int_{(-L_1)} + \cdots + \int_{L_n} + \int_{C_n} + \int_{-L_n}\right) f(z)\,dz$$

$$= \int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

\#

Cor: (Principle of deformation of paths)

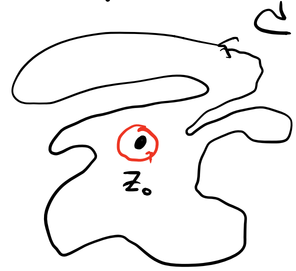
Let C_1 & C_2 be positively oriented simple closed contours, where C_2 is interior to C_1. If f is analytic in the closed region consisting of C_1, C_2, and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

(Pf: By Thm $\int_{C_1} f(z)\,dz + \int_{(-C_2)} f(z)\,dz = 0$.)

eg: Let C = any positively oriented simple closed contour surrounding the point z_0.

Then $\int_C \frac{dz}{z-z_0} = 2\pi i$.



Pf: Choose $C_\varepsilon : z = z_0 + \varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$,

with $\varepsilon > 0$ small enough such that $\{ |z - z_0| \le \varepsilon \}$ interior to C.

Then by Corollary

$$\int_C \frac{dz}{z-z_0} = \int_{C_\varepsilon} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\, d\theta}{\varepsilon e^{i\theta}}$$

$$= i \int_0^{2\pi} d\theta = 2\pi i.$$

※