Denote $B = \{f \in C[a,b] : -f \in A\}$

Then f crosses some lines

$\Leftrightarrow \quad f \in A \cup B$

Hence $C[a,b] \setminus Z = A \cup B$

So we only need to show that A_n is nowhere dense, $\forall n$, by proving

(1) A_n is closed $\forall n$, and

(2) $C[a,b] \setminus A_n$ is dense.

Pf of (1) Let $\{f_k\}$ be a seq. in A_n and

$$f_k \to f \text{ in } (C[a,b], d_\infty)$$

Since $f_k \in A_n$, $\exists\ \alpha_k \in [-n,n]$ and

$$x_k \in [a,b]$$

s.t. $$\begin{cases} (f_k)_{-\alpha_k}(t) \le (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k - \frac{1}{n}, x_k) \\ (f_k)_{-\alpha_k}(t) \ge (f_k)_{-\alpha_k}(x_k), & \forall t \in (x_k, x_k + \frac{1}{n}) \end{cases}$$

$(t \in [a,b])$

By passing to subseq., we may assume

$$x_k \to x_0 \in [a,b]$$
$$\alpha_k \to \alpha_0 \in [-n,n]$$

Then $(f_k)_{-\alpha_k}(t) \le (f_k)_{-\alpha_k}(x_k)$, $\forall t \in (x_k - \frac{1}{n}, x_k)$

$\Leftrightarrow \quad f_k(t) - \alpha_k t \le f_k(x_k) - \alpha_k x_k, \quad \forall \in (x_k - \frac{1}{n}, x_k)$

Now $\forall t \in (x_0 - \frac{1}{n}, x_0)$, $\exists\ k_0 \ge 0$ s.t.

$t \in (x_k - \frac{1}{n}, x_k)$, $\forall k \ge k_0$ (since $x_k \to x_0$)

Then $f_k \to f$ in $(C[a,b], d_\infty)$, $\alpha_k \to \alpha_0$, $x_k \to x_0$

we have $f(t) - \alpha_0 t \leq f(x_0) - \alpha_0 x_0$ (by letting $k \to +\infty$)

Since $t \in (x_0 - \frac{1}{n}, x_0)$ is arbitrary, we've proved

$$f_{-\alpha_0}(t) \leq f_{-\alpha_0}(x_0), \quad \forall\, t \in (x_0 - \tfrac{1}{n}, x_0]$$

Similarly, we can prove

$$f_{-\alpha_0}(t) \geq f_{-\alpha_0}(x_0), \quad \forall\, t \in (x_0, x_0 + \tfrac{1}{n})$$

Hence $f \in A_n$, $\therefore$ A_n is closed.

<u>Pf of (2)</u> Let $B^\infty_\varepsilon(f) \subset C[a,b]$ be a metric ball.

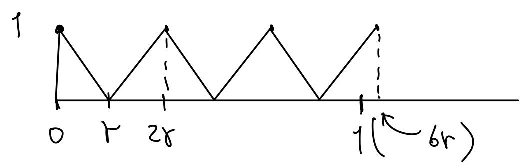
If $f \notin A_n$, then $B^\infty_\varepsilon(f) \cap (C[a,b] \setminus A_n) \neq \emptyset$.

If $f \in A_n$, by Weierstrass Approximation Theorem,

$\exists$ polynomial p s.t. $\|p - f\|_\infty < \varepsilon/3$.

Define $g(x) = p(x) + \frac{\varepsilon}{3}\varphi(x) \in C[a,b]$

where φ is the restriction to $[a,b]$ of the jig-saw function of period $2r$ satisfying $0 \leq \varphi \leq 1$, and slope of the graph of φ is $\pm\frac{1}{r}$ ($r > 0$, to be determined) (except the finitely many non-differentiable points)



Then $\|g - f\|_\infty \leq \|g - p\|_\infty + \|p - f\|_\infty \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

$\Rightarrow$ $g \in B^\infty_\varepsilon(f)$

$\Rightarrow \quad g \in V_\varepsilon(f)$.

Suppose that $g \in A_n$

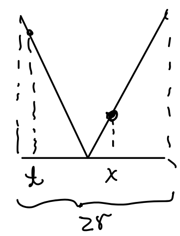
then $\exists\, x \in [a,b]$, $\alpha \in [-n,n]$ s.t

$$\begin{cases} g_{-\alpha}(t) \le g_{-\alpha}(x), & t \in (x-\frac{1}{n}, x) \\ g_{-\alpha}(t) \ge g_{-\alpha}(x), & t \in (x, x+\frac{1}{n}) \end{cases}$$

If $\varphi(x) \in [0, \frac{1}{2}]$, then consider $\forall\, t \in (x-\frac{1}{n}, x)$

$$p(t) + \frac{\varepsilon}{3}\varphi(t) - \alpha t \le p(x) + \frac{\varepsilon}{3}\varphi(x) - \alpha x$$

$$\Rightarrow \quad \varphi(x) - \varphi(t) \ge \frac{3\alpha}{\varepsilon}(x-t) - \frac{3}{\varepsilon}(p(x) - p(t))$$



By the property of φ, $\exists\, t$ with $0 < x-t < 2r$

s.t. $\quad \varphi(x) - \varphi(t) \le -\frac{1}{2}$

Consider $\quad r < \min\{\frac{1}{n}, \frac{\varepsilon}{12(L+n)}\}$, where L = Lip. const. of p.

By $r < \frac{1}{n}$, $t \in (x-\frac{1}{n}, x)$ s.t.

$$-\frac{1}{2} \ge \frac{3\alpha}{\varepsilon}(x-t) - \frac{3}{\varepsilon}(p(x) - p(t))$$

$$\Rightarrow \quad 1 \le \frac{6|\alpha|}{\varepsilon}|x-t| + \frac{6}{\varepsilon}L|x-t|$$

$$\le \frac{12(L+n)}{\varepsilon} r < 1,$$

which is a contradiction.

Hence $\varphi(x) \in [\frac{1}{2}, 1]$.

Then consider $\forall t \in (x, x+\frac{1}{n})$

$$p(t) + \frac{\varepsilon}{3}\varphi(t) - \alpha t \geq p(x) + \frac{\varepsilon}{3}\varphi(x) - \alpha x$$

$$\Rightarrow \quad \varphi(t) - \varphi(x) \geq \frac{3\alpha}{\varepsilon}(t-x) - \frac{3}{\varepsilon}(p(t) - p(x))$$

By the property of φ, $\exists t$ with $0 < t - x < 2r$

s.t. $\varphi(t) - \varphi(x) \leq -\frac{1}{2}$

Since $r < \frac{1}{n} \Rightarrow t \in (x, x+\frac{1}{n})$ s.t.

$$-\frac{1}{2} \geq \frac{3\alpha}{\varepsilon}(t-x) - \frac{3}{\varepsilon}(p(t) - p(x))$$

$$\Rightarrow \quad 1 \leq \frac{12}{\varepsilon}(k+n)r < 1 \quad \text{as before.}$$

Again, it is a contradiction.

Therefore $g \notin A_n$. And $B_\varepsilon^\infty(f) \cap [C[a,b] \setminus A_n] \neq \phi$.

This completes the proof of the Theorem. ∦

Def: A function $f: [a,b] \to \mathbb{R}$ is said to be <u>nowhere monotonic</u> if $\exists$ no interval $[c,d] \subset [a,b]$ on which f is monotonic

Cor: The set of continuous nowhere monotonic functions is a residual set in $C[a,b]$, & hence dense in $C[a,b]$.

Pf: If $f \in C[a,b]$ is monotonic on some interval $[c,d]$, then $LX \equiv b$ with $b \in (f(c), f(d))$ crosses f if $f(d) > f(c)$ (or $b \in (f(d), f(c))$ if $f(c) > f(d)$)

If $f(c) = f(d)$, then $f \equiv$ const. on $[c,d]$. Clearly many lines cross f. Hence f monotonic on some interval $\Rightarrow f \in C[a,b] \setminus Z$

Since $C[a,b] \setminus Z$ is of 1st category, any subset of $C[a,b] \setminus Z$ is also of 1st category.

$\Rightarrow$ set of cts functions monotonic on some interval is of 1st category

$\therefore$ Set of cts nowhere monotonic functions is a residual. ※

Remark: The Thm can be used to prove Thm 4.13 too.

Another application of Baire Category Theorem

> **Thm 4.14** Every basis of an infinite dimensional Banach space consists of uncountably many vectors.

Pf: Let V be a Banach space.

Suppose on the contrary that V has a countable basis $\mathcal{B} = \{w_j\}_{j=1}^{\infty}$.

Then $V = \bigcup_{n=1}^{\infty} W_n$

where $W_n = \operatorname{span}\{w_1, \cdots, w_n\}$

Claim 1: W_n has empty interior

Pf: Since V is of infinite dimensional,

$$V \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$

$$\Rightarrow \quad \{v \in V : \|v\| = 1\} \setminus W_n \neq \emptyset, \quad \forall n = 1, 2, \cdots$$

$\therefore$ one can find $v_0 \in V \setminus W_n$ such that $\|v_0\| = 1$.

Then $\forall\, w \in W_n$ and $\varepsilon > 0$,

$$w + \varepsilon v_0 \in B_\varepsilon(w) \cap (V \setminus W_n)$$

$$\Rightarrow \quad B_\varepsilon(w) \cap (V \setminus W_n) \neq \emptyset$$

$\therefore$ W_n has empty interior.

Claim 2 W_n is closed, $\forall n = 1, 2, \cdots$

Pf: Let $\{v_j\}_{j=1}^{\infty}$ be a seq in W_n and converges to some $v_0 \in V$.

Note that
$$T : W_n \to \mathbb{R}^n$$
$$\sum_{j=1}^{n} a_j w_j \mapsto (a_1, \cdots, a_n)$$

is a vector space isomorphism.

And hence the norm in V, $\left|\sum_{j=1}^{n} a_j w_j\right|_V$ gives a norm on $\mathbb{R}^n$
$$\|(a_1, \cdots, a_n)\| = \left|\sum_{j=1}^{n} a_j w_j\right|_V .$$

Since any two norms on $\mathbb{R}^n$ are equivalent, (HW8) $\|(a_1, \cdots, a_n)\|$ is equivalent to standard Euclidean norm $|(a_1, \cdots, a_n)| = \sqrt{a_1^2 + \cdots + a_n^2}$

$\Rightarrow \exists\ C_1, C_2 > 0$ s.t.
$$|v|_V \leq C_1 |Tv| \leq C_2 |v|_V , \quad \forall v \in W_n$$

Since $v_\ell \to v_0$ in V, $\{v_\ell\}$ is Cauchy in $(V, |\cdot|_V)$

$\therefore \forall \varepsilon > 0, \exists\ \ell_0 \geq 0$ s.t.
$$|v_\ell - v_k|_V < \varepsilon , \quad \forall \ell, k \geq \ell_0$$
$$\Rightarrow \quad |Tv_\ell - Tv_k| \leq \frac{C_2}{C_1} |v_\ell - v_k| < \frac{C_2}{C_1} \varepsilon , \quad \forall \ell, k \geq \ell_0$$

$\Rightarrow \{Tv_\ell\}$ is Cauchy in $\mathbb{R}^n$ (with standard metric)

By completeness of $\mathbb{R}^n$, $\exists\ a^* = (a_1^*, \cdots, a_n^*) \in \mathbb{R}^n$ s.t. $|Tv_\ell - a^*| \to 0$ as $\ell \to \infty$.

Let $v^* = T^{-1} a^* = \sum_{j=1}^{n} a_j^* w_j \in W_n$,

we have

$$|v_\ell - v^*|_V \le C_1 |Tv_\ell - a^*| \to 0 \quad \text{as } \ell \to \infty$$

By uniqueness of limit $v_0 = v^* \Rightarrow v_0 \in W_n$.

$\therefore$ W_n is closed. This proves Claim 2.

By Claims 1 & 2, W_n is nowhere dense and $V = \bigcup_{n=1}^{\infty} W_n$ is of 1^{st} category. But V is complete, this is impossible. Hence any basis of V cannot be countable. #

Final Exam:

Ch1 Fourier Series

- Riemann-Lebesgue Lemma
- pointwise and uniform convergence
- Weierstrass Approximation Theorem
- L^2-convergence (mean convergence)
- Parserval's Identity

Ch2 Metric Spaces

- Basic notations
- Open and Closed Sets
- Interior, closure & boundary
- Elementary Inequalities for Functions (Pf omitted)
  (Young's, Höldor's, Minkowski's)

Ch3 <u>Contraction Mapping Principle</u>

- Completeness
- Fixed points & Contraction
- Perturbation of Identity
- Inverse Function Theorem (Implicit Function Thm)
- Picard-Lindelöf Thm (IVP in ODE)

Ch4 <u>Space of Continuous Functions</u>

- Ascoli's Theorem
  (equicontinuity, uniform bddness, precompact)
- Arzela's Theorem
- Cauchy-Peano Thm (IVP in ODE)
- Baire Category Thm
  (nowhere dense, 1st category, residual)
- Applications of Baire Category Thm
  (eg nowhere differentiable continuous functions & etc)