MATH3060 HW6 Due date: Nov 19, 2021 (at 12:00 noon)

1. Let (X, d), (Y, ρ) be metric spaces and E be a compact subset of X. Show that if a map $f: E \to Y$ is continuous, then it is uniform continuous. (See definition in Q1 of HW5.)

2. Show that the family of functions $\{f_n(x) = x^n\}_{n=1}^{\infty}$ is equicontinuous on the interval $[0, \delta]$ for any $\delta < 1$, but not equicontinuous on the interval $[0, 1]$.

3. Define $T: C[0,1] \to C[0,1]$ by

$$(Tf)(x) = \cos^2 x + \int_0^x \frac{f(t)}{1+f^2(t)}\, dt.$$

Show that the image set $T(C[0,1])$ is precompact in $C[0,1]$.

4. Let $K(x,t) \in C([a,b] \times [a,b])$ and $g \in C[a,b]$. Prove the following:

(a) $\forall \lambda \in \mathbb{R}$, $T_\lambda : C[a,b] \to C[a,b]$ defined by

$$(T_\lambda f)(x) = \lambda \int_a^b K(x,t) f(t)\, dt + g(x)$$

is uniformly continuous.

(b) If $\mathcal{E} \subset C[a,b]$ is bounded, then $T_\lambda(\mathcal{E})$ is precompact in $C[a,b]$.

(End)