MATH3060 HW4 Due date: Oct 21, 2021 (at 12:00 noon)

1. Let X be a nonempty set, d be the discrete metric on X. Show that

   (a) $B_\varepsilon(x_0) = \{x_0\}$ $\forall\ 0<\varepsilon<1$.

   (b) Every subset of X is both open and closed.

2. Let $X = C[0,1]$, $d_1(f,g) = \int_0^1 |f-g|$, and $d_\infty(f,g) = \|f-g\|_\infty$ (for $f,g \in X$). Let $B'_\varepsilon(f)$ be an open ball in $(C[0,1], d_1)$ and $B^\infty_\varepsilon(f)$ be an open ball in $(C[0,1], d_\infty)$.

   (a) Is $B^1_\varepsilon(f)$ open in $(C[0,1], d_\infty)$?

   (b) Is $B^\infty_\varepsilon(f)$ open in $(C[0,1], d_1)$?

3. (a) Show that on $l_2 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^\infty |x_i|^2 < \infty,\ x_i \in \mathbb{R} \}$,

$$d_2(x,y) = \left(\sum_{i=1}^\infty |x_i - y_i|^2 \right)^{1/2}, \text{ for } x,y \in l_2,$$

   is a metric

   (b) Show that $H = \{ x = (x_1, x_2, \cdots) : |x_i| \le 1/i,\ x_i \in \mathbb{R} \}$ is a closed subset of l_2.

4. Let $[a,b)$ be a semi open interval in $\mathbb{R}$. Consider the metric subspace $([a,b), |x-y|)$ of $(\mathbb{R}, |x-y|)$. For $c \in (a,b)$,

(a) discuss whether $[a,c)$ is open or closed as a subset of $([a,b), |x-y|)$, and

(b) discuss whether (c,b) is open or closed as a subset of $([a,b), |x-y|)$.

(c) Find all subsets of $[a,b)$ which are both open and closed.

(End)