eg. 2.1 $(X = \mathbb{R},\ d(x,y) = |x-y|)$ is a metric space.

eg. 2.2 Let $X = \mathbb{R}^n$, $d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i-y_i)^2}$ (Euclidean metric)
for $x=(x_1,\cdots,x_n)$ & $y=(y_1,\cdots,y_n) \in \mathbb{R}^n$.
Then $(\mathbb{R}^n, d_2)$ is a metric space. (Proof omitted, Ex!)

eg. 2.3 Let $X = \mathbb{R}^n$, $d_1(x,y) = \sum_{i=1}^{n} |x_i-y_i|$

$$d_\infty(x,y) = \max_{i=1,\cdots,n} |x_i-y_i|$$

Then $(\mathbb{R}^n, d_1)$ & $(\mathbb{R}^n, d_\infty)$ are metric spaces.

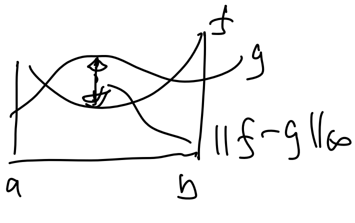
Generalization of egs 2.2 & 2.3 to function spaces:

eg 2.4 Let $C[a,b] = \{$ (real) continuous functions on $[a,b]\}$

$\forall\ f, g \in C[a,b]$, define

$$d_\infty(f,g) = \|f-g\|_\infty = \max\{|f(x)-g(x)| : x \in [a,b]\}$$

Then $(C[a,b], d_\infty)$ is a metric space (Ex!)



Similarly, one can define

$$d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx$$

It is also easy to verify that $(C[a,b], d_1)$ is a metric space. (Ex!)

The natural generalization of the Euclidean metric to $C[a,b]$ is

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2}$$

(M1) & (M2) are clear for d_2 (because f, g cts.).

Note that $d_2(f,g) = \|f-g\|_2$ (as in Fourier series)

$\Rightarrow$ d_2 satisfies (M3) (Ex! Using Cauchy-Schwarz to prove $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$)

$\therefore$ $(C[a,b], d_2)$ is a metric space.

eg 2.5 On $X = R[a,b] = \{$ Riemann integrable functions on $[a,b] \}$.

d_1 is still defined

$$d_1(f,g) = \int_a^b |f-g|$$

However, (M1) is not satisfied:

$$d_1(f,g) = 0 \Leftrightarrow f = g \text{ almost everywhere}$$
$$\not\Rightarrow f = g \text{ (at every point)}$$

$\therefore$ d_1 is not a metric on $R[a,b]$.

To overcome this, we consider $X = R[a,b]/\sim$

where "$\sim$" is an equivalent relation on $R[a,b]$

defined by $f \sim g \Leftrightarrow f = g$ almost everywhere.

(Check: "$\sim$" is an equivalent relation.)

Then elements of $R[a,b]/\sim$ can be represented as

$[f]$ or $\bar{f} = \{ g \in R[a,b] : g = f$ almost everywhere $\}$

Now define $\tilde{d}_1$ on $R[a,b]/\sim$ by

$$\tilde{d}_1(\bar{f}, \bar{g}) = d_1(f,g)$$

<u>Check:</u> $\tilde{d}_1$ is well-defined, i.e. indep. of the choice of representatives f & g:

$\forall f_1 \in \bar{f}, g_1 \in \bar{g}$.

Then $d_1(f_1, g_1) = \int |f_1 - g_1|$

$$\leq \int \cancelto{0}{|f_1 - f|} + \int |f-g| + \int \cancelto{0}{|g - g_1|}$$

$$= d_1(f, g)$$

Similarly $d_1(f,g) \leq d_1(f_1, g_1)$

$\therefore \quad d_1(f,g) = d_1(f_1, g_1)$.

Then it is straight forward to verify that $(R[a,b]/\sim, \tilde{d}_1)$ is a metric space.

Similarly for $(R[a,b]/\sim, \tilde{d}_2)$ is a metric space & note that $\tilde{d}_2$ is the L^2- distance defined before.

<u>Def</u>: A <u>norm</u> $\|\cdot\|$ is a function on a <u>real vector space</u> X to $[0,\infty)$ s.t. $\forall x, y \in X$ & $\alpha \in \mathbb{R}$

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$".

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$

(N3) $\|x+y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a <u>normed space</u>.

And $d(x,y) \stackrel{\text{def}}{=} \|x-y\|$ is called the <u>metric induced</u> by the norm $\|\cdot\|$.

(Ex: Show that $d(x,y) = \|x-y\|$ is a metric with the property $d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$, $\forall \alpha \in \mathbb{R}$)

eg: $\|x\|_2 = (\sum x_i^2)^{1/2}$, $\|x\|_1 = \sum |x_i|$

$\|x\|_\infty = \max\{|x_1|, \cdots, |x_n|\}$ are norms on $\mathbb{R}^n$

$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$, $\|f\|_1 = \int_a^b |f|$

$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$ are norms on $C[a,b]$

We've seen: "norm" $\xrightarrow{\text{induces}}$ "metric"

eg X = non-empty set

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

discrete metric on X

(Ex: Check this is a metric)

- X not necessary a vector space, so d is not induced by norm.
- Even X is a vector space:

$$\left.\begin{matrix} 1 \\ 0 \end{matrix}\right\} = d(\alpha x, \alpha y) = |\alpha|\, d(x,y) = \begin{cases} |\alpha| \\ 0 \end{cases}$$

Contradiction for $|\alpha| \neq 1$ (for $x \neq y$)

Def: Let (X,d) be a metric space. Then for any non-empty $Y \subset X$, $(Y, d|_{Y\times Y})$ is called a metric subspace of (X,d).

Notes: (i) metric subspace is a metric space.

(ii) We simple write (Y,d) for $(Y, d|_{Y\times Y})$

(iii) A metric subspace of a normed space needs not be a normed space, unless it is a vector subspace.