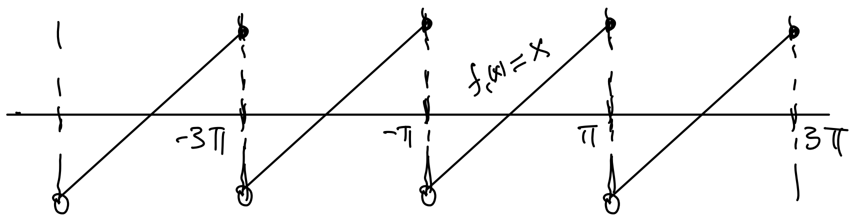
Thm 15 Let f be a 2π-periodic function integrable on $[-\pi,\pi]$. Suppose that f is Lipschitz continuous at x. Then $\{S_n f(x)\}$ converges to $f(x)$ as $n \to \infty$.

(Pf: later at the end of this section)

eg of application

Recall $f_1(x) = x$ on $[-\pi,\pi]$



Fourier series $\quad x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

It is clear that $f_1(x)$ is Lip. cts at any $x \in (-\pi,\pi)$

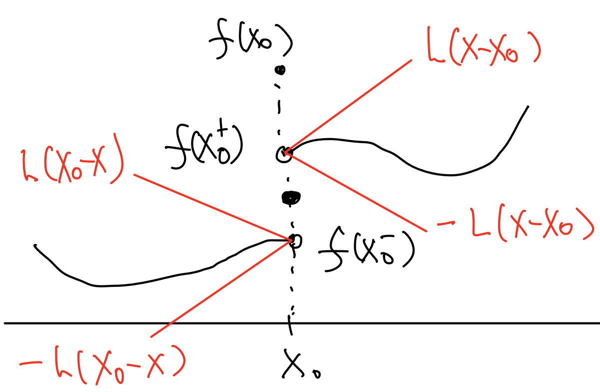
$$\therefore \lim_{N\to+\infty} 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin nx = x, \quad \forall x \in (-\pi,\pi)$$

On the other hand, $\tilde{f}_1$ is discontinuous at $x = \pm\pi$

and we've seen that (eg 1.1)

$$\tilde{f}_1(\pm\pi) \neq 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Jump discontinuity



Thm 1.6 Let f be a 2π-periodic function integrable on $[-\pi, \pi]$.
Suppose that for $x_0 \in [-\pi, \pi]$.

(i) $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ right-hand limit

$f(x_0^-) = \lim_{x \to x_0^-} f(x)$ left-hand limit

both exist.

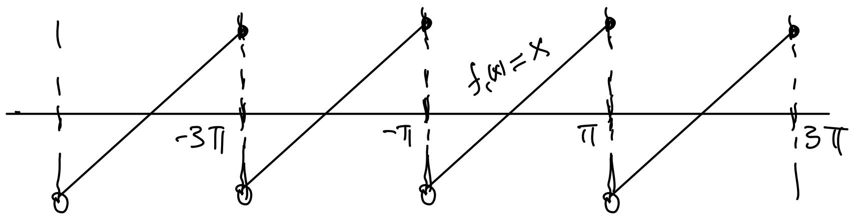
(ii) $\exists\, L > 0$ and $\delta > 0$ such that

$$\begin{cases} |f(x) - f(x_0^+)| \leq L(x - x_0), & 0 < x - x_0 < \delta \\ \& \quad |f(x) - f(x_0^-)| \leq L(x_0 - x), & 0 < x_0 - x < \delta \end{cases}$$

Then $S_n f(x_0) \longrightarrow \dfrac{f(x_0^+) + f(x_0^-)}{2}$ as $n \to +\infty$

(Pf: Omitted)

<u>eg of application</u> $f_1(x) = x$ on $[-\pi, \pi]$



At $x_0 = \pi$, $\tilde{f}_1(x)$ is discontinuous.

(i) $\tilde{f}(\pi^+) = \lim_{x \to \pi^+} \tilde{f}_1(x) = -\pi$

$\tilde{f}(\pi^-) = \lim_{x \to \pi^-} \tilde{f}_1(x) = \pi$

(ii) For $0 < x - x_0 < \frac{\pi}{2}$ (i.e. $0 < x - \pi < \frac{\pi}{2} = \delta$)

we have $|\tilde{f}(x) - \tilde{f}(\pi^+)| = |\tilde{f}(x - 2\pi) - (-\pi)|$ $L = 1$

$= |x - 2\pi + \pi| = x - \pi \leq L(x - \pi)$

Similarly for $0 < x_0 - x < \frac{\pi}{2}$

Hence conditions of Thm 1.6 are satisfied

$\Rightarrow$ Fourier series $\underset{\underset{0}{\parallel}}{S_n f(\pi)} \longrightarrow \frac{f(\pi^+)+f(\pi^-)}{2} = \frac{-\pi+\pi}{2} = 0$.

#

Next we turn to "uniform" convergence and need

Def: A function f defined on $[a,b]$ is called to <u>satisfy a Lipschitz condition</u> if $\exists L>0$ such that
$$|f(x)-f(y)| \le L|x-y|, \quad \forall x,y \in [a,b].$$

<u>Notes</u>: (1) $L>0$ is independent of $x,y\in[a,b]$, a kind of "uniform" Lip condition.

(2) f satisfies a <u>Lip. condition</u> $\Rightarrow$ f is Lip. cts at every point on $[a,b]$.

<u>eg</u>: If $f\in C^1[a,b] \Rightarrow |f(x)-f(y)| = \left|\int_x^y f'(t)dt\right|$
$$\le M|y-x|, \quad \forall x,y\in[a,b].$$

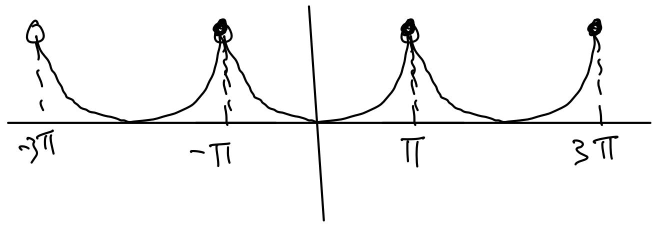
where $M = \sup_{[a,b]} |f'|$.

But $f(x)=|x|$ satisfies a Lip condition, but not C^1.

<u>Thm 1.7</u> Let f be a 2π-periodic function satisfying a Lipschitz condition. Then its Fourier Series converges <u>uniformly</u> to f itself.

(Pf: Omitted)

eg of application $f_2(x) = x^2$ on $[-\pi, \pi]$



$\tilde{f}_2$ satisfies a Lip. condition (Check! (Ex))

$$\Rightarrow \quad \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nx \quad \text{converges uniformly to}$$

$f_2(x) = x^2$ on $[-\pi, \pi]$.

(Ex: Put $x=0$ and get $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$)

Proof of Thm 1.5

Let f be Lip cts. at a point $x_0 \in [-\pi, \pi]$.

Step 1 $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0 + z)\, dz$$

where

$$D_n(z) = \begin{cases} \frac{\sin (n+\frac{1}{2}) z}{2\pi \sin \frac{1}{2} z}, & \text{if } z \neq 0 \\ \frac{2n+1}{2\pi}, & \text{if } z = 0, \end{cases}$$

is called the Dirichlet kernel.

Pf:

$$(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{k=1}^{n}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ky\,dy\right)\cos kx_0 + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ky\,dy\right)\sin kx_0\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}\left(\cos ky\cos kx_0 + \sin ky\sin kx_0\right)\right] f(y)\,dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(y-x_0)\right] f(y)\,dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n}\cos kz\right] f(z+x_0)\,dz \qquad \left(\begin{array}{c} z = y - x_0\ \& \\ 2\pi\text{-periodic}\end{array}\right)$$

Since $\frac{1}{2} + \sum_{k=1}^{n}\cos kz = \frac{\sin(n+\frac{1}{2})z}{2\sin(\frac{1}{2}z)}$ for $z \neq 0$ (check $z=0$ is correct too)

(Ex: Calculate $e^{-in\theta} + \cdots + 1 + \cdots + e^{in\theta}$
using $1 + z + \cdots + z^k = \frac{z^{k+1}-1}{z-1}$)

$$(S_n f)(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z} f(x_0+z)\,dz$$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\,dz$$

※

Step 2 (Properties of $D_n(z)$)

(1) $\int_{-\pi}^{\pi} D_n(z)\,dz = 1$

(2) $D_n(z)$ is even, cts, 2π-periodic on $[-\pi,\pi]$
and $D_n\left(\frac{2k\pi}{2n+1}\right) = 0$ for $k = -n, \cdots, 0, \cdots, n$

(3) $\max_{[-\pi,\pi]} D_n(z) = D_n(0) = \frac{2n+1}{2\pi}$

(4) $\forall\ 0<\delta<\pi/2$, $\int_0^{\delta} |D_n(z)|\,dz \to +\infty$ as $n \to +\infty$.

Pf (1) Easy : by integrating $\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos kz\right)dz$

(2) & (3) are easy exercises.

(4) Let $0<\delta<\pi/2$

Then $\forall\, n\in\mathbb{N}$, $\exists\, N\in\mathbb{N}$ s.t.

$$N<\frac{(n+\frac{1}{2})\delta}{\pi}\leq N+1$$

Clearly $N\to+\infty$ as $n\to+\infty$.

Now

$$\int_0^{\delta}|D_n(z)|\,dz=\int_0^{\delta}\frac{|\sin(n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|}\,dz$$

$$=\int_0^{(n+\frac{1}{2})\delta}\frac{|\sin t|}{2\pi\,|\sin\frac{t}{2n+1}|}\,\frac{2\,dt}{2n+1}\qquad t=(n+\tfrac{1}{2})z$$

$$=\frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta}\frac{|\sin t|}{t}\cdot\frac{\left(\frac{t}{2n+1}\right)}{\left|\sin\frac{t}{2n+1}\right|}\,dt$$

$$\geq\frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta}\frac{|\sin t|}{t}\,dt\qquad\left(\text{Using }\frac{|\sin x|}{x}<1\text{ for }0<x\right)$$

$$\geq\frac{1}{\pi}\int_0^{\pi N}\frac{|\sin t|}{t}\,dt$$

$$=\frac{1}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi}\frac{|\sin t|}{t}\,dt$$

$$=\frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi}\frac{|\sin s|}{s+(k-1)\pi}\,ds\qquad(s=t-(k-1)\pi)$$

$$\geq\frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi}\frac{|\sin s|}{k\pi}\,ds\qquad\left(s+(k-1)\pi=t\leq k\pi\right)$$

$$=\frac{1}{\pi^2}\left(\int_0^{\pi}|\sin s|\,ds\right)\sum_{k=1}^{N}\frac{1}{k}=\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k}$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, and $N\to\infty$ as $n\to\infty$, we have $\lim_{n\to\infty}\int_0^{\delta}|D_n(z)|\,dz=+\infty$ ※

<u>Step 3</u> Splitting $(S_n f)(x_0) - f(x_0) = I + II$ into integrals concentrated in $[-\delta, \delta]$ & (essentially) outside $[-\delta, \delta]$.

By (1) in step 2, $f(x_0) = \int_{-\pi}^{\pi} D_n(z) f(x_0)\, dz$

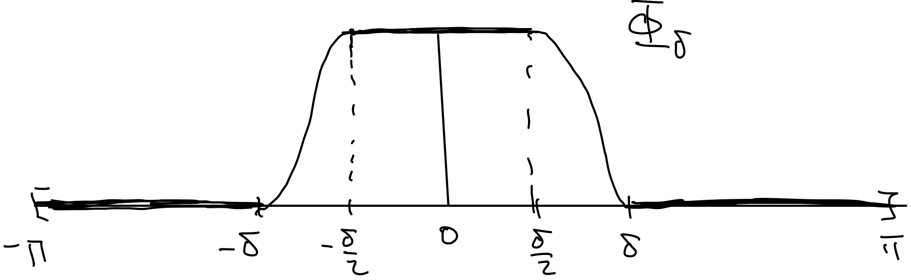
$$\therefore \ (S_n f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} D_n(z) [f(x_0+z) - f(x_0)]\, dz$$

Let Φ_δ be a "cut-off" function s.t.

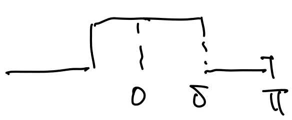
(i) Φ_δ is cts & $0 \le \Phi_\delta \le 1$

(ii) $\Phi_\delta(t) \equiv 1$ for $|t| \le \frac{\delta}{2}$

(iii) $\Phi_\delta(t) \equiv 0$ for $|t| \ge \delta$.



(This is used because of the proof of Thm 1.7 is in mind.



is enough if we only want to prove Thm 1.5)

Then $(S_n f)(x_0) - f(x_0)$

$$= \int_{-\pi}^{\pi} D_n(z) [f(x_0+z) - f(x_0)]\, dz$$

$$= \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z) [f(x_0+z) - f(x_0)]\, dz$$

$$+ \int_{-\pi}^{\pi} (1 - \Phi_\delta(z)) D_n(z) [f(x_0+z) - f(x_0)]\, dz$$

$$= I + II,$$

where
$$I = \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z) [f(x_0+z) - f(x_0)] dz$$
$$= \int_{-\delta}^{\delta} \Phi_\delta(z) D_n(z) [f(x_0+z) - f(x_0)] dz$$

$$II = \int_{-\pi}^{\pi} (1-\Phi_\delta(z)) D_n(z) [f(x_0+z) - f(x_0)] dz$$
$$= \left(\int_{-\pi}^{-\delta/2} + \int_{\delta/2}^{\pi}\right) (1-\Phi_\delta(z)) D_n(z) [f(x_0+z) - f(x_0)] dz$$

Step 4: $\exists L>0$ and $\delta_2>0$ such that
$$|I| \leq \frac{4\delta L}{\pi}, \quad \forall\, 0<\delta<\delta_2.$$

Pf: By Lip cts. at x_0, $\exists L>0$ & $\delta_0>0$ s.t.
$$|f(x_0+z) - f(x_0)| \leq L|z|, \quad \forall\, |z|<\delta_0$$
Since $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, $\exists \delta_1>0$ s.t.
$$\left|\frac{\sin\frac{z}{2}}{\frac{z}{2}}\right| > \frac{1}{2}, \quad \forall |z|<\delta_1$$
Therefore, for $\delta_2 = \min\{\delta_0, \delta_1\} > 0$
$$\frac{|f(x_0+z) - f(x_0)|}{|\sin\frac{z}{2}|} \leq \frac{L|z|}{\frac{1}{2}|\frac{z}{2}|} = 4L, \quad \forall\, |z|<\delta_2$$
Hence $\forall\, 0<\delta<\delta_2$, we have
$$|I| \leq \int_{-\delta}^{\delta} |\Phi_\delta(z)|\, |D_n(z)|\, |f(x_0+z) - f(x_0)|\, dz$$
$$= \int_{-\delta}^{\delta} |\Phi_\delta(z)| \frac{|\sin(n+\frac{1}{2})z|}{2\pi|\sin\frac{z}{2}|} |f(x_0+z) - f(x_0)|\, dz$$
$$\leq \int_{-\delta}^{\delta} 1 \cdot \frac{1}{2\pi} \cdot 4L\, dz = \frac{4\delta L}{\pi}$$
\#

<u>Step5</u> $\forall \varepsilon>0$, $\exists \delta>0$ & $n_0>0$ s.t.

$$\frac{4\delta L}{\pi} < \frac{\varepsilon}{2} \quad \text{and} \quad |\mathrm{II}| < \frac{\varepsilon}{2}, \quad \forall n \ge n_0.$$

Pf: $\forall \varepsilon>0$, we take $\delta = \min\left\{\frac{\varepsilon\pi}{8L}, \delta_2\right\} > 0$ (L & δ_2 as in step 4)

Then $\frac{4\delta L}{\pi} < \frac{\varepsilon}{2}$,

and for this fixed $\delta>0$,

$$\mathrm{II} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\Phi_\delta(z)\right)\cdot\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{\sin\frac{z}{2}}\left(\sin nz\cos\frac{z}{2}+\cos nz\sin\frac{z}{2}\right)dz$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2\sin\frac{z}{2}}\cos\frac{z}{2}\right]\sin nz\,dz$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]\cancel{\sin\frac{z}{2}}}{2\cancel{\sin\frac{z}{2}}}\right]\cos nz\,dz$$

$$= b_n(F_1)+a_n(F_2)$$

where $F_1(z) = \dfrac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2\sin\frac{z}{2}}\cos\frac{z}{2}$

$F_2(z) = \dfrac{(1-\Phi_\delta(z))\left[f(x_0+z)-f(x_0)\right]}{2}$

$F_2(z)$ is clearly integrable on $[-\pi,\pi]$.

For $F_1(z)$, note that $1-\Phi_\delta(z)=0$ on $[-\frac{\delta}{2},\frac{\delta}{2}]$

$|\sin\frac{z}{2}| \ge \sin\frac{\delta}{4} > 0$ for $\frac{\delta}{2}\le|z|\le\pi$

$\Rightarrow F_1(z)$ is also integrable on $[-\pi,\pi]$.

Therefore Riemann-Lebesgue Lemma implies

$$\left.\begin{array}{l} b_n(F_1) \\ a_n(F_2) \end{array}\right\} \to 0 \quad \text{as } n \to \infty$$

$\therefore \exists\, n_0 > 0$ s.t. $\quad |b_n(F_1)|$ & $|a_n(F_2)| < \frac{\varepsilon}{4}, \ \forall n \geq n_0$

$$\Rightarrow \quad |\mathrm{II}| \leq |b_n(F_1)| + |a_n(F_2)| < \frac{\varepsilon}{2} \quad \#$$

<u>Final Step</u> By Steps 3, 4 & 5, we have

$\forall \varepsilon > 0, \ \exists\, n_0 > 0$ s.t.

$$\begin{aligned} |(S_nf)(x_0) - f(x_0)| &= |\mathrm{I} + \mathrm{II}| \\ &\leq |\mathrm{I}| + |\mathrm{II}| \leq \frac{4\delta L}{\pi} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq n_0 \end{aligned}$$

$\therefore (S_nf)(x_0) \to f(x_0)$ as $n \to \infty$. #