§ 1.2 Riemann-Lebesgue Lemma

Recall: A step function on $[-\pi, \pi]$ is a function of the form

$$S(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

where (i) $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

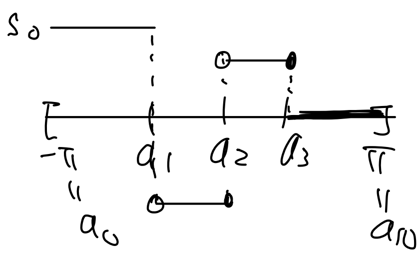
$I_0 = [a_0, a_1]$

$-\pi = a_0 < a_1 < \cdots < a_{N-1} < a_N = \pi$

(ii) For a set E, $\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$

is the characteristic function for E.

(iii) $s_j \in \mathbb{R}$, $j = 0, \cdots, N-1$.



Lemma 1.2 For every step function s integrable on $[-\pi, \pi]$, $\exists$ constant $C > 0$ (indep. of n, but depends on s) such that $|a_n(s)|, |b_n(s)| \leq \frac{C}{n}$, $\forall n \geq 1$
where $a_n(s)$, $b_n(s)$ are Fourier coefficients of s.

Pf: Let $S(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}(x)$

then for $n \geq 1$

$$\pi a_n(s) = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{N-1} s_j \chi_{I_j}(x) \right) \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \frac{1}{n}\left[\sin(n a_{j+1}) - \sin(n a_j)\right]$$

$$\Rightarrow \quad |a_n(s)| \le \frac{1}{n} \cdot \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j| = \frac{C}{n}, \quad C = \frac{2}{\pi}\sum_{j=0}^{N-1} |s_j|$$

Similarly for $|b_n(s)| \le \frac{C}{n}$, $\forall n \ge 1$. ※

Lemma 1.3 Let f be integrable on $[-\pi, \pi]$. Then $\forall \varepsilon > 0$, $\exists$ a step function $s(x)$ such that

(i) $s \le f$ on $[-\pi,\pi]$, &

(ii) $\int_{-\pi}^{\pi} (f - s) < \varepsilon$

Pf: f (Riemann) integrable

$\Rightarrow$ f can be approximated from below by Darbour lower sums.

i.e. $\forall \varepsilon > 0$, $\exists$ partition $a_0 = -\pi < a_1 \cdots < a_N = \pi$

s.t. $$\int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) < \varepsilon$$

where $m_j = \inf\{ f(x) : x \in [a_j, a_{j+1}] \}$

Define the step function

$$s(x) = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) \quad (\text{i.e. } s_j = m_j)$$

with $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

$I_0 = [a_0, a_1]$.

Then $s \le f$ & $\int_{-\pi}^{\pi} s(x)\,dx = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j)$

$\therefore$ $\int_{-\pi}^{\pi} (f - s) < \varepsilon$. ※

Now we can prove

Thm 1.1 (Riemann-Lebesgue Lemma)
The Fourier coefficients of a 2π-periodic function f integrable on $[-\pi,\pi]$ converge to 0 as $n\to+\infty$.

Pf: $\forall\, \varepsilon>0$, Lemma 1.3 $\Rightarrow$ $\exists$ step function s s.t.

$$s\le f \quad \& \quad \int_{-\pi}^{\pi}(f-s)<\frac{\varepsilon}{2}$$

Then by Lemma 1.2, $\exists\, n_0>0$ s.t.

$$|a_n(s)|<\frac{\varepsilon}{2}, \quad \forall\, n\ge n_0 \qquad \left(n_0=\left[\frac{2C}{\varepsilon}\right]+1, \text{ where } C \text{ as in Lemma 1.2} \right)$$

Therefore
$$|a_n(f)-a_n(s)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi}(f-s)(x)\cos nx\,dx\right|$$
$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} f-s < \frac{\varepsilon}{2\pi} \qquad (\text{as } f\ge s)$$

Hence
$$|a_n(f)| \le |a_n(s)| + |a_n(f)-a_n(s)|$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2\pi} < \varepsilon, \quad \forall\, n\ge n_0$$

$\therefore\ a_n(f)\to 0$ as $n\to+\infty$.

Similarly for $b_n(f)$. ※

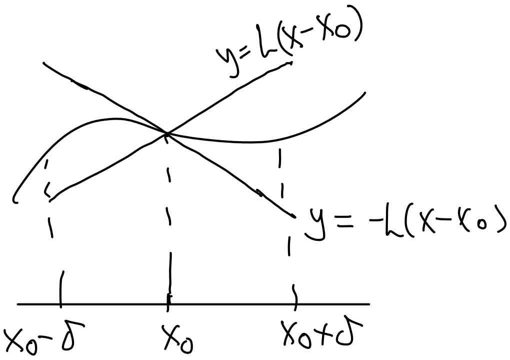
§1.3 Convergence of Fourier Series

Terminology: For $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

we denote $(S_n f)(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$

the n-th partial sum of the Fourier Series of f.

Def: Let f be a function on $[a,b]$. Then f is called Lipschitz continuous at $x_0 \in [a,b]$ if $\exists\, L>0$ & $\delta>0$ such that $$\lvert f(x)-f(x_0)\rvert \le L\lvert x-x_0\rvert, \quad \forall\, \lvert x-x_0\rvert<\delta \quad (x\in[a,b])$$



Notes (1) Both L & δ may depend on x_0

(2) If f is Lipschitz continuous at $x_0 \in [a,b]$ & f is bounded on $[a,b]$.

then $\exists\, L'>0$ (L' may depends on x_0) s.t.

$$\lvert f(x)-f(x_0)\rvert \le L'\lvert x-x_0\rvert, \quad \forall\, x\in[a,b].$$

Pf: By defn, f Lip. cts. at x_0

$\Rightarrow \exists\, L>0,\ \delta>0$ s.t

$$\lvert f(x)-f(x_0)\rvert \le L\lvert x-x_0\rvert, \quad \forall\, \lvert x-x_0\rvert<\delta$$

If $\lvert x-x_0\rvert \ge \delta$,

then $\dfrac{|x-x_0|}{\delta} \geq 1$

$$\Rightarrow |f(x)-f(x_0)| \leq |f(x)|+|f(x_0)| \leq 2M \quad \text{where } M = \sup_{[a,b]}|f|,$$

$$\leq \frac{2M}{\delta}|x-x_0|$$

Hence $|f(x)-f(x_0)| \leq \begin{cases} L|x-x_0|, & |x-x_0|<\delta \\ \frac{2M}{\delta}|x-x_0|, & |x-x_0|\geq\delta \end{cases}$

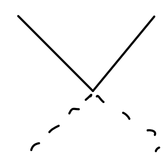
$$\Rightarrow |f(x)-f(x_0)| \leq L'(x-x_0), \quad \forall x \in [a,b],$$

where $L' = \max\{L, \frac{2M}{\delta}\} > 0$ ✗

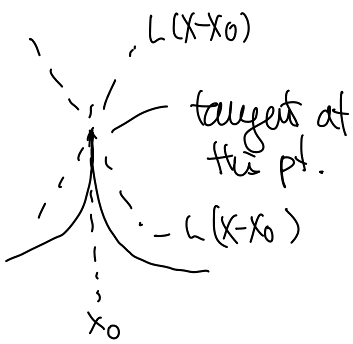
eg: $f \in C^1[a,b]$ (continuously differentiable on $[a,b]$)

$\Rightarrow$ f is Lip. cts. at every $x_0 \in [a,b]$.

On the other hand $f(x)=|x|$ is Lip. cts. at $x=0$, but not differentiable (Ex!)



eg:



this graph gives a cts function at x_0, but not Lip. cts at x_0

More precisely $f(x)=|x|^{\alpha}$ with $0<\alpha<1$ is not Lip. cts. at $x=0$.