§83 Zeros and Poles

Thm : Suppose that

(a) $p(z)$ and $q(z)$ are analytic at a point z_0;

(b) $p(z_0) \neq 0$ and $q(z)$ has a zero of order m at z_0.

Then $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order m at z_0

Pf : By (b) $q(z) = (z-z_0)^m g(z)$, $g(z)$ analytic & nonzero at z_0

$$\therefore \quad f(z) = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{\left(p(z)/g(z)\right)}{(z-z_0)^m}$$

with $\phi(z) = \dfrac{p(z)}{g(z)}$ analytic at z_0 &

$$\phi(z_0) = \frac{p(z_0)}{g(z_0)} \neq 0 . \quad \#$$

Thm2 : Let $p(z)$, $q(z)$ be analytic at z_0. If

$$p(z_0) \neq 0, \quad q(z_0) = 0 \quad \& \quad q'(z_0) \neq 0 .$$

Then z_0 is a simple pole of $f(z) = \dfrac{p(z)}{q(z)}$

and $$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} .$$

Pf: Recall the proof in Thm 1,

$$f(z) = \frac{p(z)}{(z-z_0)g(z)} \qquad g(z) \text{ analytic \& nonzero at } z_0.$$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{g(z_0)} \quad \text{(by Thm in previous sections.)}$$

Note that

$$q(z) = \cancel{q(z_0)}^{\,0} + q'(z_0)(z-z_0) + \frac{q''(z_0)}{2}(z-z_0)^2 + \cdots$$

$$= (z-z_0)\left[q'(z_0) + \frac{q''(z_0)}{2}(z-z_0) + \cdots\right]$$

$$\therefore \quad g(z) = q'(z_0) + \frac{q''(z_0)}{2}(z-z_0) + \cdots$$

$$\therefore \quad g(z_0) = q'(z_0) .$$

eg: $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

Let $p(z) = \cos z$, $q(z) = \sin z$

Then $\forall\, z = n\pi$, $n \in \mathbb{Z}$, $q(n\pi) = 0$

and $q'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$

$\therefore$ all $z = n\pi$ are simple poles of $\cot z$

and $$\operatorname{Res}_{z=n\pi} \cot z = \frac{p(n\pi)}{q'(n\pi)} = \frac{\cos(n\pi)}{\cos(n\pi)}$$

$$= 1 . \quad \#$$

eg $$f(z) = \frac{z - \sinh z}{z^2 \sinh z}$$

Consider the pole $z = \pi i$

Let $p(z) = z - \sinh z$ & $q(z) = z^2 \sinh z$

Then $p(\pi i) = \pi i \neq 0$

$q(\pi i) = 0$

$q'(z) = 2z \sinh z + z^2 \cosh z$

$\therefore \quad q'(\pi i) = (\pi i)^2 \cosh(\pi i) = (-\pi^2)(-1)$

$= \pi^2 \neq 0$

$\therefore$ $z = \pi i$ is a simple pole of $f(z)$ &

$$\operatorname{Res}_{z=\pi i} f(z) = \operatorname{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z}$$

$$= \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}$$

$\#$

eg $f(z) = \dfrac{z}{z^4+4}$ (Reading Ex!)

$z = 1+i$ is a simple pole of $f(z)$ and

$$\operatorname{Res}_{z=1+i} \frac{z}{z^4+4} = -\frac{i}{8}$$

\#

§84 Behavior of the functions near isolated singular points

(a) Removable Singular Points

Thm 1 If z_0 is a removable singular point of f, then f is bounded and analytic in $0 < |z - z_0| < \varepsilon$, for some $\varepsilon > 0$.

Thm 2 (Riemann's Thm)
Suppose that f is bounded and analytic in $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$. Then either f is analytic at z_0 or f has a removable singular point at z_0.

(Pfs: Omitted)

(b) Essential Singular Point

Thm3 (Casorati - Weierstrass Thm)

Suppose that z_0 is an essential singularity of f, and w_0 be any complex number. Then $\forall \varepsilon > 0$ and $\delta > 0$, $\exists\ z \in \{0 < |z - z_0| < \delta\}$ such that

$$|f(z) - w_0| < \varepsilon,$$

Remark : This implies if z_0 is an essential singularity of f, then $\forall\ w_0 \in \mathbb{C}$, $\exists\ z_n \to z_0$ certain $z_n \neq z_0$ such that $f(z_n) \to w_0$ as $n \to \infty$.

(Pf: Omitted)

(c) Pole of order m

Thm 4 If z_0 is a pole of f, then

$$\lim_{z \to z_0} f(z) = \infty .$$

(Pf : Omitted)