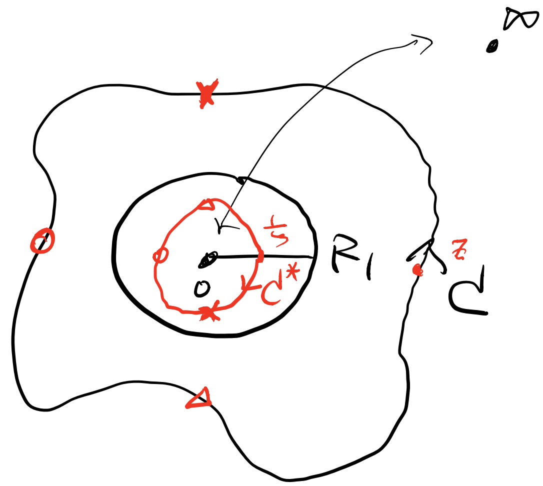
§77 Residue at Infinity

Suppose $f(z)$ is analytic in $R_1 < |z| < \infty$ and $C = z(t)$ simple closed contour in $R_1 < |z| < \infty$ surrounding $z=0$. (positively oriented)



Then $\displaystyle\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

Change of variable $\zeta = \frac{1}{z}$,

Then C^*: $\zeta(t)$ is a negative oriented simple closed contour surrounding 0

and $\zeta'(t) = -\dfrac{z'(t)}{z^2(t)} \Longrightarrow z'(t) = -\dfrac{\zeta'(t)}{\zeta^2(t)}$.

$$\therefore \int_C f(z)dz = \int_a^b f\left(\frac{1}{\zeta(t)}\right)\left(-\frac{\zeta'(t)}{\zeta^2(t)}\right)dt$$

$$= -\int_a^b \left[\frac{1}{\zeta^2(t)} f\left(\frac{1}{\zeta(t)}\right)\right]\zeta'(t)\,dt$$

$$= -\int_{C'^*} g(\zeta)\,d\zeta = \int_{-C'^*} g(\zeta)\,d\zeta$$

where $g(\zeta) = \frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)$ analytic in $0 < |\zeta| < \frac{1}{R_1}$

Since $-C'^*$ is positively oriented simple closed contour surrounding $\zeta = 0$, we have

$$\int_{C'} f(z)\,dz = 2\pi i \operatorname{Res}_{\zeta=0} g(\zeta).$$

$$= 2\pi i \operatorname{Res}_{\zeta=0}\left[\frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right)\right].$$

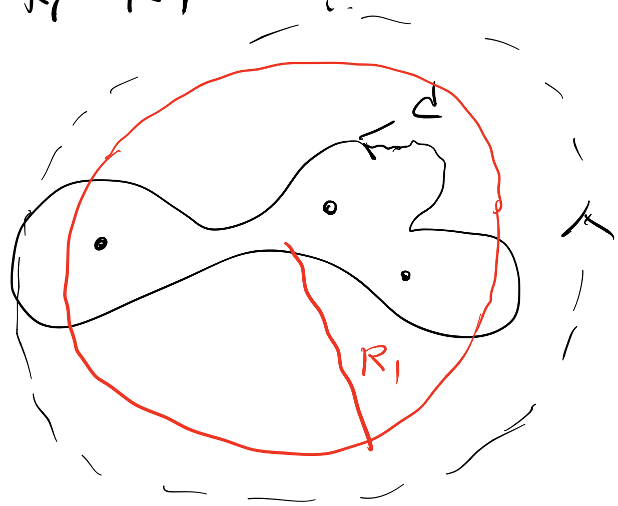
Therefore, we have

Thm If f is analytic everywhere in the plane except for a finitely many singular points interior to a positively oriented simple closed contour C', then

$$\boxed{\int_{C'} f(z)\,dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].}$$

Pf: By assumption, $\exists\, R_1 > 0$ s.t.

f is analytic in $R_1 < |z| < \infty$.



Then by deformation principal

$$\int_C f(z)\,dz = \int_{|z|=R_1+\delta} f(z)\,dz$$

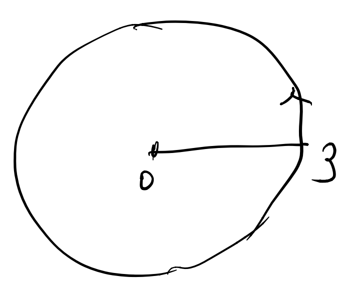
$$= 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

by the observation right before the theorem. ✗

eg: Evaluate $\displaystyle\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz$

for $C: |z|=3$ positively oriented.

Soln: $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$



analytic in $2 < |z| < \infty$

& $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z^2} \cdot \dfrac{\left(\frac{1}{z}\right)^3\left(1-\frac{3}{z}\right)}{\left(1+\frac{1}{z}\right)\left(1+\frac{2}{z^4}\right)}$

$$= \frac{1}{z^2} \cdot \frac{1}{z^3} \cdot \frac{z-3}{z} \cdot \frac{z}{z+1} \cdot \frac{z^4}{z^4+2}$$

$$= \frac{z-3}{z(1+z)(2+z^4)}$$

$$= \frac{1}{z}(-3)\left(1-\frac{z}{3}\right)\frac{1}{1+z} \cdot \frac{1}{2\left(1+\frac{z^4}{2}\right)}$$

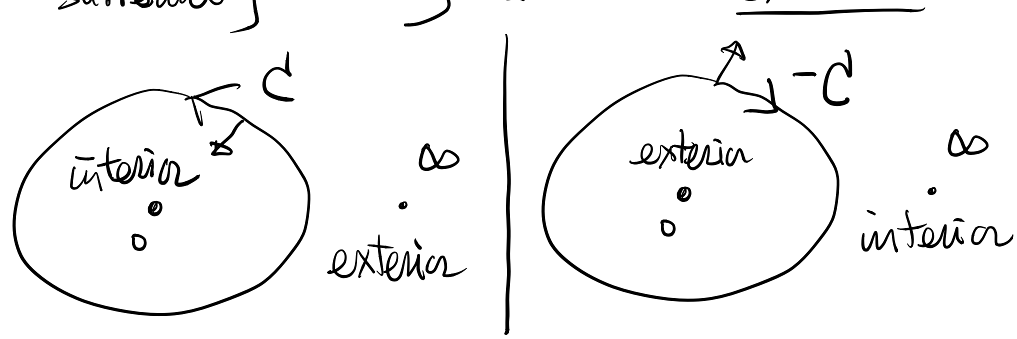
$$= -\frac{3}{2} \cdot \frac{1}{z}\left(1-\frac{z}{3}\right)\left(1-z+\cdots\right)\left(1-\frac{z^4}{2}+\cdots\right)$$

$$= -\frac{3}{2} \cdot \frac{1}{z} + \cdots$$

$$\therefore \quad \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2}$$

Hence $\displaystyle\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i\left(-\frac{3}{2}\right) = -3\pi i$

<u>Note</u>: For a positively oriented C in $R_1 < |z| < \infty$ surrounding $z=0$, $z=\infty$ is <u>exterior</u> to C



Therefore, we may think of $z=\infty$ is interior to $-C'$.

$$\therefore \frac{1}{2\pi i}\int_{-C'} f(z)dz = -\operatorname{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]$$

can be defined as the residue of f at $z=\infty$.

Then $$\boxed{\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]}.$$

§78 The Three Types of Isolated Singular Points

If f is analytic in $0<|z-z_0|<R_2$. Then f has a Laurent series expansion about z_0.

Def: (1) The portion

$$\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

is called the principal part of f at z_0.

(2) Removable singular points

If $b_1=b_2=\cdots=b_n=\cdots=0$, then z_0 is called a removable singular point of f.

(3) Essential Singular Points

If there are infinitely many nonzero b_n in the principal part, then z_0 is called an essential singular point of f.

(4) Poles of order m

If $\exists\, m \geq 1$ such that $b_m \neq 0$, but

$$b_{m+1} = b_{m+2} = \cdots = 0,$$

then z_0 is called a pole of order m of f. And a pole of order $m=1$ is called a simple pole.