§ 72 Uniqueness of Series Representation

We have used the following theorem many times:

Thm 1 If a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z-z_0|=R$, then it is the Taylor series expansion for f in powers of $z-z_0$.

Pf: By assumption

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0|<R).$$

Consider $g(z) = \frac{1}{2\pi i} \frac{1}{(z-z_0)^{k+1}}$ $\quad k=0,1,3,3,\cdots$

on a $\underbrace{\text{simple}}_{\text{positively oriented}}$ closed contour C surrending z_0 and interior to $|z-z_0|=R$. Then Cauchy integral formula $\Rightarrow$

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{k+1}} = \int_C \left[g(z) \sum_{n=0}^{\infty} a_n (z-z_0)^n \right] dz$$

$$= \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz \qquad \left(\begin{array}{l}\text{by Thm 1 in} \\ \text{§71}\end{array}\right)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{2\pi i} \int_C \frac{(z-z_0)^n}{(z-z_0)^{k+1}} \, dz$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{2\pi i} \int_C \frac{dz}{(z-z_0)^{k-n+1}}$$

Note that $\displaystyle \int_C \frac{dz}{(z-z_0)^{k-n+1}} = \begin{cases} 2\pi i, & \text{if } k=n \\ 0, & \text{if } k\neq n \end{cases}$

Hence $\displaystyle \frac{f^{(k)}(z_0)}{k!} = \underbrace{\frac{a_k}{2\pi i} \cdot 2\pi i}_{n=k \text{ term}} + \underbrace{0}_{\text{all other terms}}$

$$\therefore \quad a_k = \frac{f^{(k)}(z_0)}{k!} \qquad (k=0,1,2,3,\cdots)$$

#

Similarly, one has

Thm2 If

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges to $f(z)$ in some $R_1 < |z-z_0| < R_2$, then it is the Laurent series expansion for f about the point z_0. (Proof Omitted)

§73 Multiplication and Division of Power Series

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$

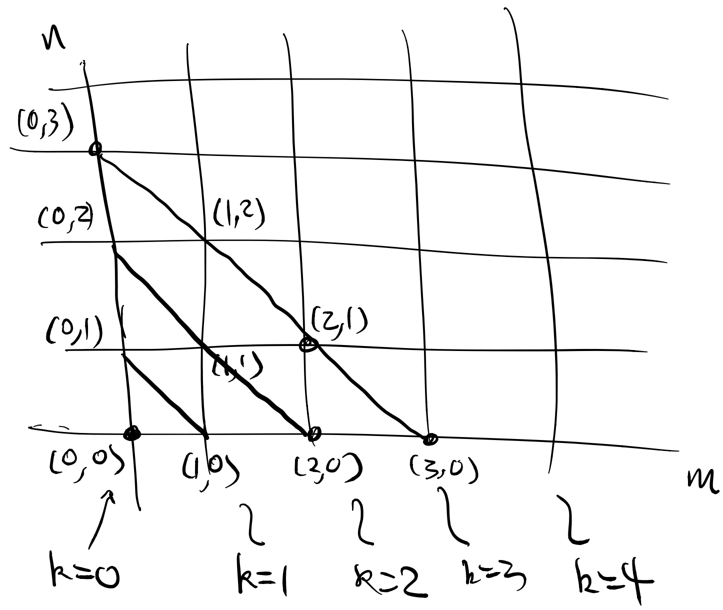
Then by renaming the index for $g(z)$ & we see

$$f(z)g(z) = \left[\sum_{n=0}^{\infty} a_n (z-z_0)^n\right]\left[\sum_{m=0}^{\infty} b_m (z-z_0)^m\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m (z-z_0)^{m+n}$$

Let $k = m+n$, then

$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_{k-l} b_l\right)(z-z_0)^k$$



$$\therefore \boxed{f(z)g(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n}$$

Another way to see this is by Leibniz's rule :

$$\left(f(z)g(z)\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)\, g(z)^{(n-k)}.$$

to show that
$$\frac{(fg)^{(n)}(z_0)}{n!} = \sum_{k=0}^{n} a_k b_{n-k} \quad (\text{Ex!}).$$

eg 1: $f(z) = \dfrac{\sinh z}{1+z}$ $\quad |z|<1$. (Find 1st few terms of the Taylor Series (up to z^4))

$$= (\sinh z)\left(\frac{1}{1+z}\right)$$

$$= \left(z + \frac{z^3}{3!} + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right)$$

$$= z - z^2 + z^3 - z^4 + \cdots$$
$$+ \frac{z^3}{3!} - \frac{z^4}{3!} + \cdots$$

$$= z - z^2 + \left(1+\frac{1}{6}\right)z^3 - \left(1+\frac{1}{6}\right)z^4 + \cdots$$

$$= z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \cdots \quad (|z|<1)$$

eg 2 (Division)

$$\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} = \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

Long division:

$$\begin{array}{r|l} & 1 - \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots \\ \hline 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots & 1 \\ & 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \\ \hline & -\frac{z^2}{3!} - \frac{z^4}{5!} - \cdots \\ & -\frac{z^2}{3!} - \frac{1}{(3!)^2} z^4 - \cdots \\ \hline & \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right) z^4 - \cdots \\ & \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right) z^4 - \cdots \\ \hline \end{array}$$

$$\therefore \frac{1}{\sinh z} = \frac{1}{z}\left(1 - \frac{z^2}{3!} + \frac{7}{120} z^4 - \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{120} z^3 - \cdots \qquad (0 < |z| < \pi)$$

OR:

$$\frac{1}{\sinh z} = \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots} = \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!}\left(1 + \frac{3!}{5!} z^2 + \cdots\right)}$$

$$= \frac{1}{z}\left\{1 - \frac{z^2}{3!}\left(1 + \frac{3!}{5!} z^2 + \cdots\right) + \left[\frac{z^2}{3!}\left(1 + \frac{3!}{5!} z^2 + \cdots\right)\right]^2 - \cdots\right\}$$

$$= \frac{1}{z}\left\{ 1 - \frac{1}{3!}z^2 - \frac{z^4}{5!} \cdots + \frac{z^4}{(3!)^2}\left(1 + \frac{3!}{5!}z^2 + \cdots\right)^2 + \cdots \right\}$$

$$= \frac{1}{z}\left[1 - \frac{1}{3!}z^2 - \frac{z^4}{5!} + \frac{z^4}{(3!)^2}\left(1 + \cdots\right) + \cdots \right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{6}z^2 + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4 + \cdots \right]$$

$$= \frac{1}{z} - \frac{1}{6}z + \frac{7}{120}z^3 + \cdots \qquad (0 < |z| < \pi)$$

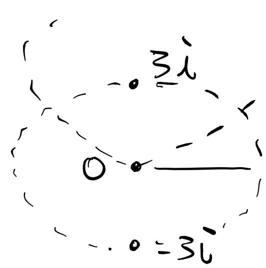
Ch6 Residues and Poles

§74 Isolated Singular Points

Def: A singular point z_0 is said to be isolated if there is a deleted ε-neighborhood $0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic.

(i.e. $\exists\, \varepsilon>0$ s.t. f is analytic in $0<|z-z_0|<\varepsilon$)

eg1: $f(z)=\dfrac{z-1}{z^5(z^2+9)}$ has isolated singular points at $z=0$ & $z=\pm 3i$:



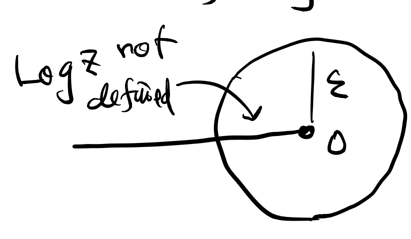
f analytic in $0<|z|<3$

$0<|z-3i|<3$

& $0<|z+3i|<3$

eg2: $z=0$ is not an isolated singular point of the principal branch of $\operatorname{Log} z$:

$\operatorname{Log} z$ is not analytic in any $0<|z|<\varepsilon$.



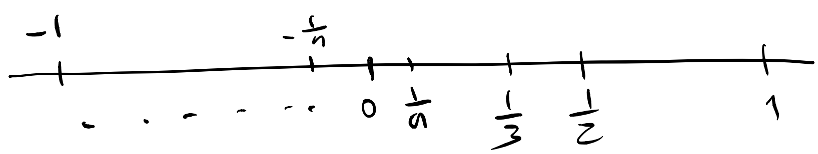
eg 3 $\quad f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

f is not defined

z singular $\iff \sin\frac{\pi}{z} = 0$ or $z = 0$

$\iff \frac{\pi}{z} = n\pi \quad (n = \pm 1, \pm 2, \cdots)$, or $z = 0$

$\iff z = \frac{1}{n}$, $n = \pm 1, \pm 2, \cdots$, or $z = 0$



For $z = \frac{1}{n}$, f is analytic in $0 < |z - \frac{1}{n}| < \frac{1}{|n(n+1)|}$

$\Rightarrow z = \frac{1}{n}$, $n = \pm 1, \cdots$, are isolated singular points.

But $z = 0$ is not isolated, since $0 < |z| < \varepsilon$ contains $\frac{1}{n}$ for some $n = \pm 1, \pm 2, \cdots$ & hence $f(z)$ is not analytic in $0 < |z| < \varepsilon$.

<u>Note</u>: If f is analytic in $R_1 < |z| < \infty$, then f is said to have an <u>isolated singular point at $z_0 = \infty$</u>.

§75 Residue

<u>Note</u>: If z_0 is an isolated singular point of f, then f has a Laurent series representation about $z = z_0$ as it is analytic in $0 < |z - z_0| < \varepsilon$.

<u>Def</u>: The <u>residue</u> of f at an isolated singular point z_0 is the coefficient $b_1 = \frac{1}{2\pi i}\int_C f(z)dz$ of the term $\frac{1}{z - z_0}$ in the Laurent expansion of about z_0, and is denoted by

$$\operatorname{Res}_{z=z_0} f(z) = b_1 .$$

(Where C is any positively oriented simple closed contour surrending z_0 & interior to $|z - z_0| = \varepsilon$.)

<u>eg1</u>: Let $f(z) = \frac{e^z - 1}{z^5}$, then

$z = 0$ is an isolated singular point of f and f is analytic in $0 < |z| < \infty$.

To find $\operatorname{Res}_{z=0} f(z)$, we expand f as follows:

$$f(z) = \frac{1}{z^5}(e^z - 1)$$

$$= \frac{1}{z^5}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) - 1\right]$$

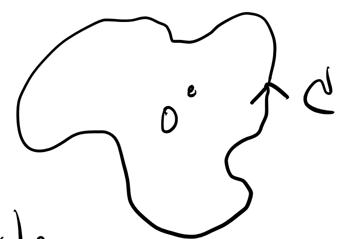
$$= \frac{1}{z^5}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)$$

$$= \frac{1}{z^4} + \cdots + \frac{1}{4!}\frac{1}{z} + \cdots$$

$\therefore$ $\operatorname{Res}_{z=0} f(z) = \frac{1}{4!} = \frac{1}{24}$.

Note: $\frac{1}{24} = \operatorname{Res}_{z=0} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz$

$\Rightarrow \int_C f(z)\,dz = \frac{\pi i}{12}$



for any positively oriented simple closed contour C. (surrending $z=0$)

eg 2 Let C: positively oriented unit circle $|z|=1$.

Show that $\int_C \cosh\left(\frac{1}{z^2}\right) dz = 0$.

Pf: $\cosh\left(\frac{1}{z^2}\right)$ analytic in $0<|z|<\infty$.

$\therefore$ $z=0$ is an isolated singular point of $\cosh\left(\frac{1}{z^2}\right)$ and hence

$$\int_C \cosh\left(\frac{1}{z^2}\right) dz = 2\pi i \operatorname{Res}_{z=0} \cosh\left(\frac{1}{z^2}\right)$$

To find $\operatorname{Res}_{z=0} \cosh\left(\frac{1}{z^2}\right)$, we consider

$$\cosh\left(\frac{1}{z^2}\right) = 1 + \frac{\left(\frac{1}{z^2}\right)^2}{2!} + \frac{\left(\frac{1}{z^2}\right)^4}{4!} + \cdots$$

$$= 1 + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{4!}\frac{1}{z^8} + \cdots \quad (0<|z|<\infty)$$

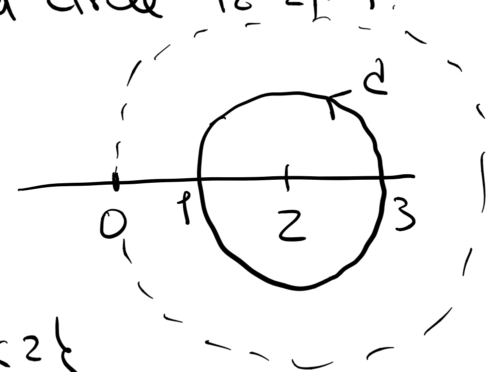
$\Rightarrow$ $\frac{1}{z}$ term is 0

$\therefore$ $\operatorname{Res}_{z=0} \cosh\left(\frac{1}{z^2}\right) = 0$

$\therefore$ $\int_C \cosh\left(\frac{1}{z^2}\right) dz = 0$ ⋕

eg3 : Evaluate $\displaystyle\int_C \frac{dz}{z(z-2)^5}$

for C : positively oriented circle $|z-2|=1$.



Soln : $\dfrac{1}{z(z-2)^5}$ is analytic

in $0<|z-2|<2$

and $C \subset \{0<|z-2|<2\}$

$\therefore$ $\displaystyle\int_C \frac{dz}{z(z-2)^5} = 2\pi i \operatorname{Res}_{z=2} \frac{1}{z(z-2)^5}.$

To find $\mathrm{Res}_{z=2} \frac{1}{z(z-2)^5}$, we consider

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5} \frac{1}{2+(z-2)}$$

$$= \frac{1}{2(z-2)^5} \frac{1}{1+\frac{z-2}{2}} \quad \text{or } \frac{|z-2|}{2}<1$$

$$= \frac{1}{2(z-2)^5}\left[1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3 + \left(\frac{z-2}{2}\right)^4+\cdots\right]$$

$$= \cdots + \frac{1}{2^5}\frac{1}{z-2}+\cdots,$$

$$\therefore \mathrm{Res}_{z=2} \frac{1}{z(z-2)^5} = \frac{1}{2^5} = \frac{1}{32}$$

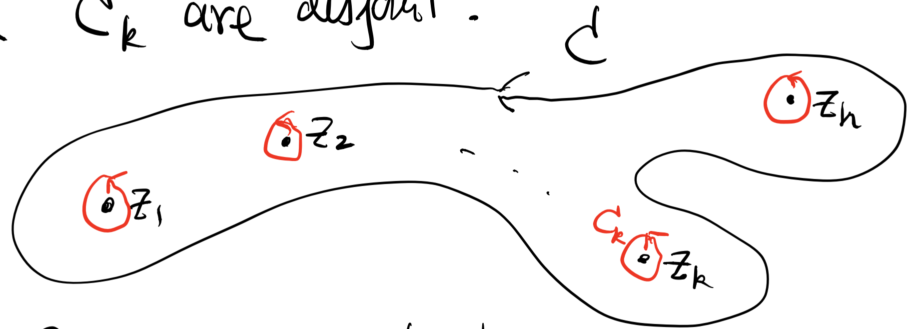
$$\therefore \int_C \frac{dz}{z(z-2)^5} = 2\pi i \frac{1}{32} = \frac{\pi i}{16}.$$

§76 Cauchy's Residue Theorem

<u>Thm</u>: Let C be a positively oriented simple closed contour. If f is analytic inside and on C <u>except</u> finitely many singular points $z_1, \ldots, z_n$ inside C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)}$$

<u>Pf</u>: Let C_k be positively oriented circle around z_k with small radius such that C_k interior to C and C_k are disjoint:



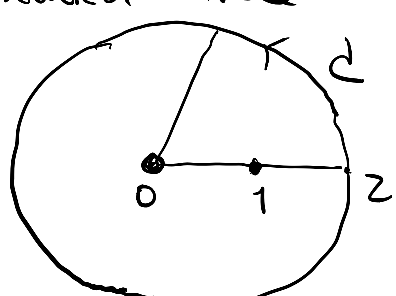
Then Cauchy–Goursat Thm $\Rightarrow$

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$= 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z). \quad \#$$

<u>eg</u>: Evaluate $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$ for

C = positively oriented circle $|z|=2$.



$z=0$ & 1 are isolated singular points inside C

$$\therefore \int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i\left(\operatorname{Res}_{z=0}\frac{4z-5}{z(z-1)} + \operatorname{Res}_{z=1}\frac{4z-5}{z(z-1)}\right)$$

Ex:
$$\begin{cases} \operatorname{Res}_{z=0}\dfrac{4z-5}{z(z-1)} = 5 \\ \operatorname{Res}_{z=1}\dfrac{4z-5}{z(z-1)} = -1 \end{cases}$$

$$\therefore \int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i\,(5-1) = 8\pi i$$

[Note: $\dfrac{4z-5}{z(z-1)} = \dfrac{5}{z} + \dfrac{(-1)}{z-1}$]