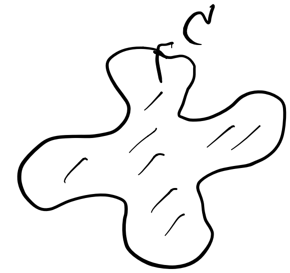
§50 Cauchy-Goursat Theorem

Thm (Cauchy-Goursat Theorem) If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

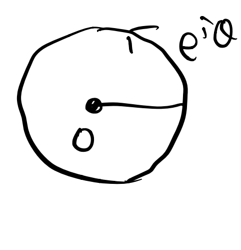


eg: C = simple closed contour in $\mathbb{C}$.
Since $\sin(z^2)$ is entire, $\sin(z^2)$ is analytic at all points interior to and on C. Hence

$$\int_C \sin(z^2)\,dz = 0.$$

eg: $f(z) = \frac{1}{z}$ is not analytic at all point interior to the unit circle $C: z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.
(not analytic at $z=0$) &

$$\int_C \frac{dz}{z} = 2\pi i \neq 0.$$



Pf of Cauchy-Goursat Theorem under an additional condition that

$f'(z)$ is continuous at all point interior to and on the simple closed contour C.
(Original Cauchy's observation.)

Let $f(z) = u(x,y) + i v(x,y)$

$C: z = z(t) = x(t) + i y(t)$, $a \le t \le b$.

Then $\int_C f(z)dz = \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy).$$

Since f' cts, u_x, u_y, v_x, v_y are cts.

Hence Green's theorem $\Rightarrow$

$$\begin{cases} \int_C udx - vdy = \iint_R (-u_y - v_x)dxdy \\ \int_C vdx + udy = \iint_R (-v_y + u_x)dxdy \end{cases}$$

where R = region enclosed by C.

Then Cauchy-Riemann eqt. $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\Rightarrow \int_C udx - vdy = 0 = \int_C vdx + udy$$ $\#$

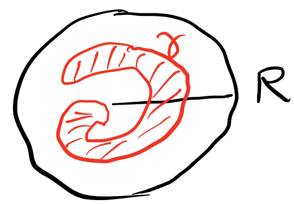
§51 Proof of the Theorem (without the additional condition) is omitted.

§52 Simply connected Domains

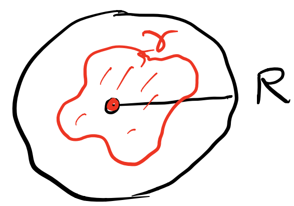
Def: A <u>simply connected domain</u> D is a domain such that every simple closed contour within it encloses only points of D.

eg:

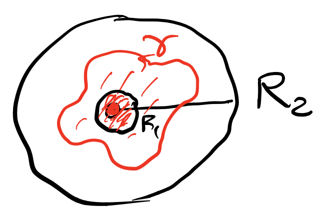
- $D = \{ z \in \mathbb{C} : |z| < R \}$ is simply-connected.



- $D \setminus \{0\} = \{ z \in \mathbb{C} : 0 < |z| < R \}$ is <u>not</u> simply-connected: $\exists\, \gamma$ encloses a region containing $0 \notin D$.

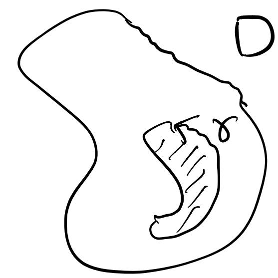


- Similarly, any <u>annulus</u> $\{ z \in \mathbb{C} : R_1 < |z| < R_2 \}$ is <u>not</u> simply-connected



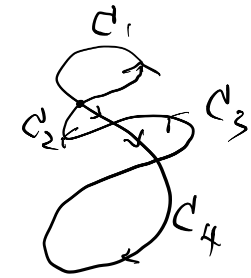
<u>Thm</u> If a function f is <u>analytic</u> throughout a simply connected domain D, then
$$\int_C f(z)\,dz = 0$$
for <u>every closed contour</u> (not necessary simple) C lying in D.

Pf: If C is simple, then the region enclosed by C is contained in D. $\therefore$ f is analytic in the region enclosed by and on C.



Cauchy-Goursat Thm $\Rightarrow$ $\int_C f(z)dz = 0$.

If C is not simple, but intersects itself a finite number of times.



Then C can be subdivided into finitely many simple closed contours. Then

$$\int_C f dz = \sum_i (\pm 1) \int_{C_i} f dz = 0 \quad \text{as } \int_{C_i} f dz = 0 \quad \forall i.$$

(Infinite many intersection: Omitted!) ✗

eg: $D = \{ |z| < 2 \}$ simply-connected

$$f(z) = \frac{\sin z}{(z^2+9)^5} \text{ is analytic in } D$$

(Since the singularities are $z = \pm 3 \notin D$)

$$\therefore \int_C \frac{\sin z}{(z^2+9)^5} dz = 0 \quad \forall \text{ closed contour in } \{|z| < 2\}.$$

Cor 1 : A function f that is analytic throughout a simply-connected domain D must have an antiderivative everywhere in D.

Cor 2 : Entire functions always possess antiderivatives.

(Pf : $\mathbb{C}$ is simply-connected.)

§53 Multiply Connected Domains

Def : A domain that is not simply-connected is said to be multiple connected.