Ch 4 Integrals

§41 Derivatives of Functions $w(t)$

$w(t) = u(t) + iv(t)$ is a cpx-valued function of a real variable $t \in (a,b)$. Then

$$\frac{d}{dt} w(t) = w'(t) = u'(t) + iv'(t)$$

(where u, v are real & imaginary parts of w.)

eg3 Mean value therem for $w'(t)$ no longer holds:

$$w(t) = e^{it}, \quad 0 \le t \le 2\pi$$

$$w'(t) = \frac{d}{dt} w(t) = \frac{d}{dt}(\cos t + i \sin t) = (-\sin t + i \cos t)$$
$$= i(\cos t + i \sin t) = ie^{it}$$

$$\Rightarrow |w'(t)| = 1, \quad \forall t.$$

$$\frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{e^{i2\pi} - e^{i\cdot 0}}{2\pi} = 0 \neq w'(c) \quad \forall c \in [0, 2\pi].$$

egs 1 & 2 (Reading exercises)

§42 Definite Integrals of Functions $w(t)$

Def: Let $w(t)$ be a cpx-valued function of a real variable $t \in [a,b]$ written as

$$w(t) = u(t) + iv(t) \qquad (u, v \in \mathbb{R})$$

Then the definite integral of $w(t)$ over $[a,b]$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the individual integrals $\int u$, $\int v$ exist.

Thus

$$\begin{cases} \mathrm{Re}\int_a^b w(t)dt = \int_a^b \mathrm{Re}[w(t)]dt \\ \mathrm{Im}\int_a^b w(t)dt = \int_a^b \mathrm{Im}[w(t)]dt \end{cases}$$

eg1: $\int_0^{\pi/4} e^{it}dt = \int_0^{\pi/4}(\cos t + i\sin t)dt$

$$= \int_0^{\pi/4}\cos t\,dt + i\int_0^{\pi/4}\sin t\,dt$$

$$= \frac{1}{\sqrt{2}} + i\left(1-\frac{1}{\sqrt{2}}\right).$$

Prop: (1) $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$ for $a \le c \le b$.

(2) Fundamental Theorem of Calculus

If $W(t) = U(t) + iV(t)$ and

$w(t) = u(t) + iv(t)$ are functions of

$t \in [a,b]$ s.t. $W'(t) = w(t)$.

Then $\displaystyle\int_a^b w(t)\,dt = W(b) - W(a) = [W(t)]_a^b$

$= W(t)\Big|_a^b$

$$\left(\int_a^b \frac{dW}{dt}(t)\,dt = W(t)\Big|_a^b \quad \text{or} \quad \int_a^b dW = W(t)\Big|_a^b \right)$$

eg 2 $\quad \dfrac{d}{dt}\left(\dfrac{e^{it}}{i}\right) = e^{it}$

$$\Rightarrow \int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big|_0^{\pi/4} = \frac{e^{i\frac{\pi}{4}} - 1}{i} = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

eg3 No mean value theorem for cpx integral:

$w(t) = e^{it}$

$$\int_0^{2\pi} w(t)\,dt = \frac{e^{it}}{i}\Big|_0^{2\pi} = \frac{e^{i2\pi} - e^{i0}}{i} = 0$$

$$\neq w(c)(2\pi - 0) \quad \forall c \in [0, 2\pi]$$

as $|w(c)| = 1 \quad \forall c \in [0, 2\pi]$.

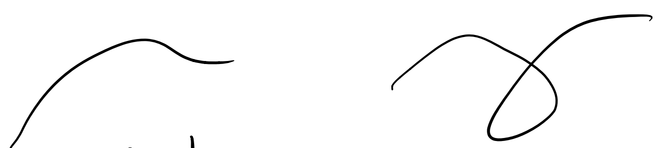
§43 Contours

Def: (1) An arc in $\mathbb{C}$ is a parametrized cts curve

$$z = z(t) = x(t) + iy(t), \quad t \in [a,b]$$

where $x(t)$ & $y(t)$ are cts functions of t.

(2) The arc C is a simple arc or Jordan arc if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$



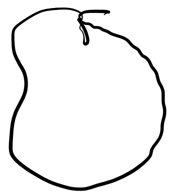
simple　　　not simple

(3) The arc C is simple closed curve or Jordan curve if C is simple except $z(b) = z(a)$.

simple　　　not simple

$x \sin \frac{1}{x}$?

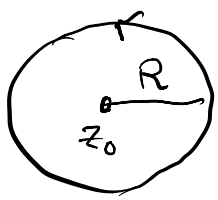
(4) A simple closed curve $C: z = z(t)$ is **positively oriented** when it is in the counterclockwise direction



positive oriented

eg1 (Reading)

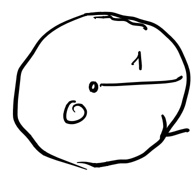
eg2: $z = z(\theta) = z_0 + Re^{i\theta}$, $0 \le \theta \le 2\pi$



positively oriented simple closed curve

circle of radius R centered at z_0 in the counterclockwise direction.

eg3: $z = e^{-i\theta}$, $0 \le \theta \le 2\pi$

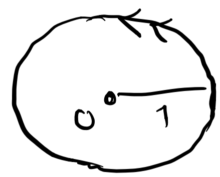


"negatively" oriented

(not positively oriented)

(clockwise direction)

eg4: $z = e^{2i\theta}$, $0 \le \theta \le 2\pi$



$z(\frac{\pi}{2}) = -1 = z(\frac{3\pi}{2})$

NOT simple as the circle is traversed **twice** in the counterclockwise direction.

Def: Change of parameters (a reparametrization)

Let C be an arc parametrized by

$$z = z(t), \quad a \le t \le b.$$

Let $\phi : [\alpha, \beta] \to [a, b]$ be a differentiable increasing function (i.e. $\phi'(\tau) > 0, \ \forall \tau \in [\alpha, \beta]$) such that $\phi(\alpha) = a$ & $\phi(\beta) = b$ ($\Rightarrow \phi$ is 1-1 & onto)

Then $$z = Z(\tau) = z \circ \phi(\tau), \quad \alpha \le \tau \le \beta$$

is called a reparametrization of $z = z(t), a \le t \le b$.

(And ϕ is called a change of parameters)