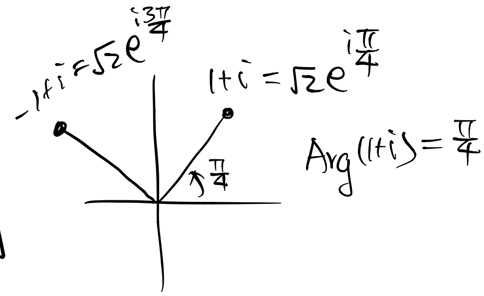
eg 4 $\quad \mathrm{Log}(1+i)^2 = 2\,\mathrm{Log}(1+i)$




Check:

$$2\,\mathrm{Log}(1+i) = 2\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]$$
$$= 2\ln\sqrt{2} + i\frac{\pi}{2}$$
$$= \ln 2 + i\frac{\pi}{2}.$$

$$(1+i)^2 = 2i = 2e^{i\frac{\pi}{2}} \qquad \mathrm{Arg}(2i) = \frac{\pi}{2} \in (-\pi, \pi].$$

$$\therefore \quad \mathrm{Log}(1+i)^2 = \ln 2 + i\frac{\pi}{2}$$
$$= 2\,\mathrm{Log}(1+i).$$

Caution: However $\mathrm{Log}(-1+i)^2 \neq 2\,\mathrm{Log}(-1+i)$.

$$2\,\mathrm{Log}(-1+i) = 2\left[\ln\sqrt{2} + i\frac{3\pi}{4}\right] = \ln 2 + i\frac{3\pi}{2}.$$

but
$$(-1+i)^2 = \left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^2 = 2e^{i\frac{3\pi}{2}}, \quad \frac{3\pi}{2} \notin (-\pi, \pi]$$

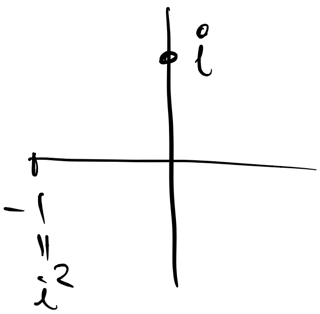
$$\therefore \quad \frac{3\pi}{2} \neq \mathrm{Arg}(-1+i)^2$$

In fact $\mathrm{Arg}(-1+i)^2 = -\frac{\pi}{2} \in (-\pi, \pi]$

$$\Rightarrow \quad \mathrm{Log}(-1+i)^2 = \ln 2 - i\frac{\pi}{2}$$
$$\neq 2\,\mathrm{Log}(-1+i).$$

eg5 : $\log(i^2) = \log(-1)$

$= (2n+1)\pi i \ , \ n \in \mathbb{Z}.$




But

$2\log i = 2\left[(2n\pi + \frac{\pi}{2})i\right], \ n \in \mathbb{Z}$

$= (4n+1)\pi i \ , \ n \in \mathbb{Z}$

$\therefore \ 2\log i \subsetneqq \log(i^2)$ (as sets.)

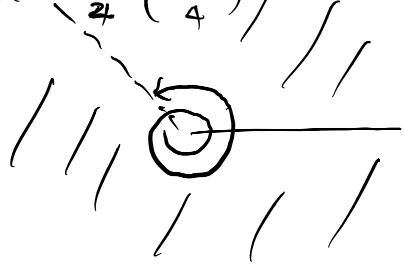
## §33 Branches and Derivatives of Logarithms

Def : For any $\alpha \in \mathbb{R}$, restriction of $\log z$ on the domain $\{re^{i\theta} : r>0, \alpha<\theta<\alpha+2\pi\}$ becomes a single-valued function

$\log z = \ln r + i\theta \ , \quad r>0, \ \alpha<\theta<\alpha+\pi$

with components $u = \ln r$ & $v = \theta$,

2 egs. of $\alpha$ $\alpha = \frac{3\pi}{4}$ $\left(\frac{11\pi}{4}\right)$



($\alpha$ could be any real number, not nec. in $(-\pi, \pi]$)

and is a called a branch of $\log z$.

For simplicity, we use the same notation $\log z$ for any branch, & use the restriction on $\theta$ to distinguish the branches:

eg
- $\log z = \ln r + i\theta$, $\frac{3\pi}{4} < \theta < \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$, $r>0$
- $\log z = \ln r + i\theta$, $-\frac{5\pi}{4} < \theta < \frac{3\pi}{4}$, $r > 0$

are different branches of $\log z$ even the domains look the same

$$\{re^{i\theta} : r>0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\} = \{re^{i\theta} : r>0, -\frac{5\pi}{4} < \theta < \frac{3\pi}{4}\}.$$

Caution: For a branch of $\log z$,

$\log z$, for a fixed $z$, is the value of the function (branch of log) at the point $z$! Not the set!
So we've to mention the branch in order to be clear!

Def: The branch defined by $\alpha = -\pi$
i.e. $r > 0$, $-\pi < \theta < \pi$

is called the Principal Branch of log and denoted by

$$\boxed{\operatorname{Log} z = \ln r + i\Theta, \quad r>0, -\pi<\Theta<\pi}$$

Notes: (1) The ray $\{\theta=\alpha\}$ is called the branch cut of the branch.

(2) The branch $\log z = \ln r + i\theta$, $r>0$, $\alpha<\theta<\alpha+2\pi$, cannot be extended continuity across the branch cut.

(3) Because of (2), we use open interval $\alpha<\theta<\alpha+2\pi$ for branch, but not $\alpha<\theta\leq\alpha+2\pi$ as in principal value $(-\pi<\Theta\leq\pi)$

Derivatives of $\log z$ (a branch of $\log z$)

Given a branch of $\log z$

$$\log z = \ln r + i\theta, \quad r>0, \ \alpha<\theta<\alpha+2\pi,$$

we have $u=\ln r$, $v=\theta$

$$\Rightarrow \quad u_r = \frac{1}{r} = \frac{1}{r}v_\theta, \quad \frac{1}{r}u_\theta = 0 = -v_r$$

$u_r, u_\theta, v_r, v_\theta$ cts & satisfy C-R eqt

on $r>0$, $\alpha<\theta<\alpha+2\pi$.

$\Rightarrow$ This branch of $\log z$ is analytic on $\{r>0, \alpha<\theta<\alpha+2\pi\}$

and $\frac{d}{dz}\log z = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

If $\alpha=-\pi$, we have

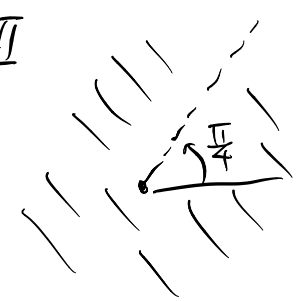
$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}, \text{ for } z=re^{i\theta}, \quad r>0,\ -\pi<\theta<\pi$$

In conclusion :

$$\boxed{\begin{array}{l} \text{For any } \alpha, \quad \frac{d}{dz}\log z = \frac{1}{z}, \text{ for } |z|>0,\ \alpha<\arg z<\alpha+2\pi. \\ \text{In particular,} \quad \frac{d}{dz}\operatorname{Log} z = \frac{1}{z}, \text{ for } |z|>0,\ -\pi<\arg z<\pi \end{array}}$$

Eg : Take a branch of $\log z$ :

$$\log z = \ln r + i\theta, \quad \frac{\pi}{4}<\theta<\frac{9\pi}{4}$$




Then $i^2 = -1 = e^{i\pi}$ in this branch

$\therefore \quad \log i^2 = i\pi$

& $\quad i = e^{i\frac{\pi}{2}}$ in this branch

$\log i = i\frac{\pi}{2}$ in this branch

$\Rightarrow \quad 2\log i = \log i^2$ in this branch.

If we take a different branch


$\log z = \ln r + i\theta, \quad r > 0$

$\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

Then $\quad i^2 = -1 = e^{i\pi}$ in this branch $\quad \pi \in (\frac{3\pi}{4}, \frac{11\pi}{4})$

$\Rightarrow \quad \log i^2 = i\pi$ in this branch.

But $\quad i = e^{i\frac{5\pi}{2}}$ in this branch $\quad \frac{5\pi}{2} \in (\frac{3\pi}{4}, \frac{11\pi}{4})$

$\Rightarrow \quad \log i = i\frac{5\pi}{2}$ in this branch

$\Rightarrow \quad 2\log i = 5\pi i$ in this branch

$\neq \log i^2$ in this branch.

## §34 Some Identities involving Logarithms

Prop: $\forall\, z_1, z_2 \in \mathbb{C}\setminus\{0\}$

$\log(z_1 z_2) = \log z_1 + \log z_2$ as sets (not branches)

Pf: $\log(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2)$

(single-valued) $\quad$ (multiple-valued)

$$= (\ln|z_1| + \ln|z_2|) + i(\arg z_1 + \arg z_2)$$
(as sets)
$$= (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2)$$
$$= \log z_1 + \log z_2 \quad \#$$

<u>Caution</u>: $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$ in general
(same for any branch of $\log z$)

## §35 The Power Function

<u>Def</u>: $\forall$ cpx number $c$, we define the <u>power function</u> by
$$z^c \overset{\text{def}}{=\!=} e^{c \log z} \quad (\text{for } z \neq 0)$$

<u>Notes</u>: (1) $z^c$ is possibly multiple-valued!

(2) For $c = n \in \mathbb{Z}$, then
$$z^n = e^{n \log z} = e^{n[\ln r + i(\Theta + 2k\pi)]}, \quad k \in \mathbb{Z}$$
$$= e^{n(\ln r + i\Theta)} e^{i 2nk\pi}, \quad k \in \mathbb{Z}$$
$$= e^{n \mathrm{Log}\, z} \text{ is single-valued.}$$

(3) For $c = \frac{1}{n}$, $n \in \mathbb{Z} \setminus \{0\}$,

$$z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}[\ln r + i(\Theta + 2k\pi)]}, \quad k \in \mathbb{Z}$$

$$= \sqrt[n]{r}\, e^{i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \cdots, n-1.$$

$$= \text{set of } n\text{-roots of } z.$$

<u>Def</u>: A <u>branch of $z^c$</u> is the function defined on the domain of a branch of $\log z$ with value given by the formula

$$z^c = e^{c \log z}, \quad r > 0, \ \alpha < \theta < \alpha + 2\pi$$

with the corresponding branch of log.

<u>Prop</u>: For any branch of $z^c$,

$$\frac{d}{dz} z^c = c z^{c-1} \quad \left(|z| > 0, \ \alpha < \arg z < \alpha + 2\pi\right)$$

<u>Pf</u>: For a branch of $\log z$, $\frac{d}{dz}\log z = \frac{1}{z}$.

$$\Rightarrow \quad \frac{d}{dz} z^c = \frac{d}{dz}\left(e^{c\log z}\right) = e^{c\log z} \frac{d}{dz}(c\log z)$$

$$= e^{c\log z} \cdot \frac{c}{z} = c e^{(c-1)\log z}$$

$$= c z^{c-1}. \quad \#$$

Def: Principal value of $z^c$, denoted by

$$\boxed{P.V.\ z^c \overset{def}{=\!=} e^{c \operatorname{Log} z}}$$

coincide with the principal brand of $z^c$,

$|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$.

---

Finally, we may also define exponential function with base $c$ by

$$c^z = e^{z \log c}$$

$$= e^{z[\ln|c| + i(\operatorname{Arg} c + 2k\pi)]}, \quad k \in \mathbb{Z}$$

is multiple-valued.

But for any value of $\log c$ ($\arg c$) is specified,

then $c^z = e^{z \log c}$ is an entire (single-valued function with $\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \log c$

$$= c^z \log c$$

the specified value.

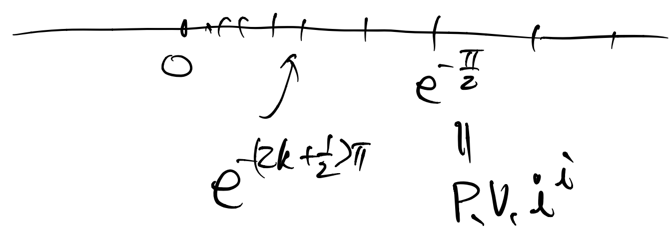
§36 Examples

eg1 : $i^i = e^{i \log i} = e^{i[\ln|i| + i(\mathrm{Arg}\, i + 2k\pi)]}$ $k \in \mathbb{Z}$

$= e^{i[i(2k+\frac{1}{2})\pi]}$ , $k \in \mathbb{Z}$

$= e^{-(2k+\frac{1}{2})\pi}$ , $k \in \mathbb{Z}$

P.V. $i^i = e^{-\frac{\pi}{2}}$




eg3 : Principal branch of $z^{2/3}$

$= e^{\frac{2}{3}\mathrm{Log}\, z} = e^{\frac{2}{3}[\ln r + i\Theta]}$ , $-\pi < \Theta < \pi$

$= \sqrt[3]{r^2}\left(\cos\frac{2\Theta}{3} + i \sin\frac{2\Theta}{3}\right)$.

eg4 : Let $z_1 = 1+i$ , $z_2 = 1-i$ , $z_3 = -1-i$

Then P.V. $z_1^i = e^{-\frac{\pi}{4} + i\ln\sqrt{2}}$ , (Ex!)

P.V. $z_2^i = e^{\frac{\pi}{4} + i\ln\sqrt{2}}$

P.V. $(z_1 z_2)^i = e^{i\ln 2} = (\text{P.V. } z_1^i)(\text{P.V. } z_2^i)$

On the other hand

$$\text{P.V. } z_3^{i} = e^{\frac{3\pi}{4} + i\ln\sqrt{2}}$$

$$\text{P.V. } (z_2 z_3)^{i} = e^{-\pi + i\ln 2}$$ (Ex!)

$$\neq (\text{P.V. } z_2^{i})(\text{P.V. } z_3^{i})$$

§40 Inverse Trigonometric & Hyperbolic functions

(1) $w = \sin^{-1} z$

<u>Soln</u>: $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$$\Rightarrow \quad e^{iw} - 2iz - e^{-iw} = 0$$

$$\Rightarrow \quad (e^{iw})^2 - 2iz(e^{iw}) - 1 = 0$$

$$\Rightarrow \quad e^{iw} = \frac{2iz + [(2iz)^2 + 4]^{1/2}}{2}$$ ← multiple-valued

$$= iz + (1 - z^2)^{1/2}$$

$$\Rightarrow \quad w = -i \log[iz + (1 - z^2)^{1/2}]$$

$$\therefore \boxed{\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]}$$

multiple-valued.

Similarly

(2) $$\boxed{\cos^{-1} z = -i \log\left[z + i(1-z^2)^{1/2}\right]}$$

(3) $$\boxed{\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}}$$

eg: $\sin^{-1}(-i) = -i \log\left[i(-i) + (1-i^2)^{1/2}\right]$

$= -i \log\left[1 + 2^{1/2}\right]$

$= -i \log(1 \pm \sqrt{2})$

$$= \begin{cases} -i\left[\ln(1+\sqrt{2}) + i\, 2k\pi\right], & k \in \mathbb{Z} \\ -i\left[\ln(\sqrt{2}-1) + i(2k+1)\pi\right], & k \in \mathbb{Z} \end{cases}$$

$$= \begin{cases} 2k\pi - i \ln(1+\sqrt{2}), & k \in \mathbb{Z} \\ (2k+1)\pi - i \ln(\sqrt{2}-1), & k \in \mathbb{Z} \end{cases}$$

$$= \begin{cases} n\pi - i \ln(1+\sqrt{2}), & n \text{ even} \\ n\pi - i \ln \frac{1}{\sqrt{2}+1}, & n \text{ odd} \end{cases}$$

$$= \begin{cases} n\pi - i \ln(1+\sqrt{2}), & n \text{ even} \\ n\pi + i \ln(1+\sqrt{2}), & n \text{ odd} \end{cases}$$

$\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1+\sqrt{2}), \quad n \in \mathbb{Z}.$

Derivatives for branches & inverse function for

Hyperbolic functions : ( Reading exercise ! )