§25 Analytic functions

Def : (1) A function $f(z)$ is analytic in an open set S if $f'(z)$ exists $\forall z \in S$.

(2) A function $f(z)$ is analytic at a point z_0 if $f(z)$ is analytic in $|z-z_0|<\varepsilon$ for some $\varepsilon>0$.

(3) An entire function is a function analytic on the entire cpx plane.

Convention : A function f said to be analytic in a set S that is not open if f is analytic in an open set S' containing S.

egs (i) $\frac{1}{z}$ is analytic in $0<|z|<+\infty$.

(ii) $f(z)=|z|^2$ is not even analytic at $z=0$ where $f'(0)$ exists (eg 2 in §23), since $\forall \varepsilon>0$, $f'(z)$ doesn't exist for $z \in \{0<|z|<\varepsilon\}$.

$\therefore$ NO ε-nbd of 0 s.t. f analytic on the ε-nbd.

(iii) Polynomials $a_0 + a_1 z + \cdots + a_n z^n$ are entire.

Simply properties :

(i) f analytic in a domain $D \Rightarrow f$ continuous in D

(ii) Analytic in $D \Rightarrow$ C-R eqt in D

(iii) $\forall$ 1st order partial derivatives exist & cts on D
+ CR eqt everywhere
$\Rightarrow$ analytic in D.

(iv) f, g analytic $\Rightarrow \begin{cases} f \pm g, \ fg \text{ analytic} \\ \frac{f}{g} \text{ analytic provided } g \neq 0. \end{cases}$

(In particular, rational function $\frac{P(z)}{Q(z)}$ is analytic in $\{z : Q(z) \neq 0\}$.)

(v) f, g analytic $\Rightarrow$ $f \circ g$ analytic &
$$(f \circ g)' = f'(g) g'$$

Thm : If $f'(z) = 0$ everywhere in a domain D, then $f(z) =$ constant throughout D.

Pf : Let $f(z) = u + iv$

then $0 = f'(z) = u_x + i v_x$

$\Rightarrow \quad u_x = v_x = 0$

C-R eqt $\Rightarrow \quad u_y = v_y = 0$ also.

Since domain D is connected, by thm in Advanced Calculus, $u \equiv u_0$, $v \equiv v_0$ constants.

$\Rightarrow \quad f(z) \equiv u_0 + i v_0$ a const. #

Def: A point z_0 is called a singular point of f if f is not analytic at z_0 but is analytic at some point in every nbd. of z_0.
(i.e. $\exists$ seq $z_n \to z_0$ s.t. f analytic at z_n, $\forall n$)

egs (i) $z=0$ is a singular point of $f(z) = \frac{1}{z}$
(f not analytic at $z=0$, but analytic in $0<|z|<\varepsilon$, $\forall \varepsilon > 0$)

(ii) $f(z) = |z|^2$ has no singular point
(not analytic everywhere.)

§26 Further Examples:

eg1 $f(z) = \dfrac{z^2+3}{(z+1)(z^2+5)}$

analytic on $\mathbb{C} \setminus \{-1, \pm i\sqrt{5}\}$

$\Rightarrow$ $-1, \pm i\sqrt{5}$ are singular points of f.

eg2 (later)

eg3: If $f = u + iv$, $\bar{f} = u - iv$

are both analytic in a domain D.

Then f = constant on D.

Pf: $\bar{f}$ analytic $\Rightarrow$ $\begin{cases} u_x = (-v)_y \\ u_y = -(-v)_x \end{cases}$

$\Rightarrow$ $\begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$

Together with analyticity of f:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases},$$

we have $\begin{cases} u_x = u_y = 0 \\ v_x = v_y = 0 \end{cases}$

$\because$ D connected $\Rightarrow$ u, v are consts.

$\Rightarrow$ f is const. ##

eg4: If f is analytic on a domain D and

$|f| \equiv$ const. on D,

then f = const. on D.

Pf: Let $|f| \equiv r_0$ a real const. on D

If $r_0 = 0$, then $f \equiv 0$ on D. We're Done.

Assume $r_0 \neq 0$, then $f(z) \neq 0$, $\forall z \in D$,

and hence $\overline{f}(z) = \dfrac{r_0^2}{f(z)}$ is analytic.

Eg3 $\Rightarrow$ $f(z) \equiv$ const. on D #

§27 Harmonic Functions

Def: A real-valued function $H = H(x,y)$ of 2-variables is said to be harmonic in a domain $D \subset \mathbb{R}^2$, if $H \in C^2(D)$ (has cts. 2nd order partial derivatives) & satisfies $\boxed{H_{xx} + H_{yy} = 0}$ (Laplace's equation)

Thm: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u, v are harmonic in D.

Pf: Sketch: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\Rightarrow \begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{cases}$$

$$\Rightarrow u_{xx} + u_{yy} = 0 \quad \#$$

egs (i) $f(z) = \sin x \cosh y + i \cos x \sinh y = u + iv$

where $\begin{cases} \cosh y = \dfrac{e^{y}+e^{-y}}{2} \\ \sinh y = \dfrac{e^{y}-e^{-y}}{2} \end{cases}$

It is easy to check

$$\begin{cases} u_x = \cos x \cosh y = v_y \\ u_y = \sin x \sinh y = -v_x \end{cases} \quad \text{cts, C-R eqts}$$

$\Rightarrow$ f is analytic.

$$\begin{cases} u_{xx} = -\sin x \cosh y \\ u_{yy} = \sin x \cosh y \end{cases}$$

$\Rightarrow \quad u_{xx} + u_{yy} = 0$. #

(ii) Reading exercise: $f(z) = \frac{1}{z^2}$ analytic in $\mathbb{C}\setminus\{0\}$

$$\Rightarrow \quad \frac{x^2-y^2}{(x^2+y^2)^2}, \quad \frac{2xy}{(x^2+y^2)^2} \quad \text{harmonic on } \mathbb{C}\setminus\{0\}.$$

(§28, 29. Postponed)

Ch3 Elementary Functions

§30 The Exponential Function

Def: The exponential function e^z or $\exp z$ is defined by

$$\boxed{\begin{array}{l} \exp z = e^z \overset{\text{def}}{=\!=} e^x e^{iy} \quad \text{for } z = x+iy \in \mathbb{C} \\ \text{where } \quad e^{iy} \overset{\text{def}}{=\!=} \cos y + i \sin y . \end{array}}$$

Notation: "$\exp z$" is a better notation in the following situation:

$$\text{for } z = \frac{1}{n}, \text{ then } \exp\frac{1}{n} = e^{\frac{1}{n}} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{n}\right)^k}{k!} \in \mathbb{R}$$

which is the positive n-root of the real number $e = 2.718\cdots$

This is in conflict with our convention that

$$z_0^{\frac{1}{n}} = \text{set of n-th roots of } z_0 \ !$$

For convenient, we will accept this exception for $e^{\frac{1}{n}}$ and interpret it as the value $\exp\left(\frac{1}{n}\right)$.

It is clear that (for $z = x+iy$)

(1) $|e^z| = e^x$, $\arg e^z = y + 2n\pi$, $n \in \mathbb{Z}$.

(2) $e^z \neq 0, \ \forall z \in \mathbb{C}$

(3) $\boxed{e^{z_1} e^{z_2} = e^{z_1 + z_2}}$ (by compound angle formula)

$$\Rightarrow \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

(4) $\boxed{\frac{d}{dz} e^z = e^z}$ $\Rightarrow$ e^z is <u>entire</u>.

Pf: $f(z) = e^z = e^x(\cos y + i \sin y)$

$$= (e^x \cos y) + i(e^x \sin y) = u + iv$$

$$\begin{cases} u_x = e^x \cos y = v_y \\ u_y = -e^x \sin y = -v_x \end{cases} \quad \text{cts, C-R eq}^{t}.$$

$\Rightarrow$ $f = e^z$ is <u>entire</u>.

& $f' = u_x + i v_x = (e^x \cos y) + i(e^x \sin y)$

$= e^z$. #

(5) $e^{z + 2\pi k i} = e^z, \ \forall k \in \mathbb{Z}$

in particular

$\boxed{e^{z + 2\pi i} = e^z}$ (ie. e^z is a periodic function with (cpx) period $2\pi i$)

$$\boxed{e^{2\pi i} = 1}$$

Lets study §37–39 first.

§37 The Trigonometric functions $\sin z$ & $\cos z$

Euler formula : $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$\Rightarrow \boxed{\begin{aligned} \cos x &= \frac{e^{ix}+e^{-ix}}{2} \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \end{aligned}}$$

Therefore, we define

Def: $\forall z \in \mathbb{C}$

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz}+e^{-iz}}{2} \\ \sin z &= \frac{e^{iz}-e^{-iz}}{2i} \end{aligned}}$$

Properties : (1) $\sin z$, $\cos z$ are entire.

$$\begin{cases} \dfrac{d}{dz}\sin z = \cos z \\ \dfrac{d}{dz}\cos z = -\sin z \end{cases} \qquad (\text{Ex!})$$

(2) $\begin{cases} \sin(-z) = -\sin z & \text{odd} \\ \cos(-z) = \cos z & \text{even} \end{cases}$

(3) $e^{iz} = \cos z + i \sin z$ generalization of Euler's formula to cpx numbers.

(4) $\begin{cases} \sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\ \cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \end{cases}$ (Ex!)

(5) $\sin^2 z + \cos^2 z = 1$ (Ex!)

(6) Real & imaginary parts of $\sin z$ & $\cos z$

$$\begin{cases} \sin z = \sin x \cosh y + i \cos x \sinh y \\ \cos z = \cos x \cosh y - i \sin x \sinh y \end{cases} \quad (z = x+iy)$$

Pf: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{(-i)}{2}\left[e^{ix-y} - e^{-ix+y}\right]$

$= \dfrac{(-i)}{2}\left[e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right]$

$= \dfrac{-i}{2}\left[-(e^{y} - e^{-y})\cos x + i(e^{y} + e^{-y})\sin x\right]$

$= \sin x \cosh y + i \cos x \sinh y$ ⁂

Similarly for $\cos z$. (Ex!)

(7) $\begin{cases} |\sin z|^2 = \sin^2 x + \sinh^2 y \\ |\cos z|^2 = \cos^2 x + \sinh^2 y \end{cases}$

Note: unbounded in y-direction as $\sinh^2 y \to +\infty$ as $y \to \pm\infty$

Pf: Hints: use (6) & $\cosh^2 y - \sinh^2 y = 1$. (Check!)

⁂