§10 Roots of Complex Numbers

(4th Edition)

Note that

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2k\pi \end{cases}$$

for some $k \in \mathbb{Z}$

Then for $z_0 = r_0 e^{i\theta_0} \neq 0$,

$$\boxed{\begin{aligned} c_k &= \sqrt[n]{r_0}\, \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, 1, 2, \cdots, n-1 \\ &= \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} \\ &\text{are all the distinct } n\text{-root of } z_0 \end{aligned}}$$

i.e. $c_k^n = z_0$, and if $w^n = z_0$, then $w = c_k$ for some $k = 0, 1, \cdots, n-1$. (Ex!)

Notations: (1) $z_0^{\frac{1}{n}}$ = set of all n-roots of z_0

$$= \{c_0, c_1, \cdots, c_{n-1}\}$$

$$= \left\{ c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} : k = 0, 1, \cdots, n-1 \right\}$$

In this notation $r_0^{\frac{1}{n}} = \left\{ c_k = \sqrt[n]{r_0}\, e^{i\frac{2k\pi}{n}} : k = 0, 1, \cdots, n-1 \right\}$

$\therefore$ $r_0^{\frac{1}{n}}$ is a set, but $\sqrt[n]{r_0}$ is a positive real number.

(2) Principal n-root:

If $z_0 = r_0 e^{i\theta_0}$ with $\theta_0 = \operatorname{Arg} z_0 \in (-\pi, \pi]$,

then $$c_0 = \sqrt[n]{r_0}\, e^{i\frac{\operatorname{Arg} z_0}{n}} \left(= \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}\right)$$

(i.e. $k=0$ in the formula)

is called the Principal n-root of z_0.

(3) Let $\omega_n = e^{i\frac{2\pi}{n}}$ $\left(= \exp\left(i\frac{2\pi}{n}\right)\right)$

Then $$\begin{cases} \omega_n^k = e^{i\frac{2k\pi}{n}} & k=0,1,\cdots,n-1 \\ \omega_n^n = 1 \end{cases}$$

Hence for $z_0 = r_0 e^{i\operatorname{Arg} z_0}$,

$$z_0^{\frac{1}{n}} = \sqrt[n]{r_0}\, e^{i\left(\frac{\operatorname{Arg} z_0}{n} + \frac{2k\pi}{n}\right)} \quad k=0,1,\cdots,n-1$$

$$= \left(\sqrt[n]{r_0}\, e^{i\frac{\operatorname{Arg} z_0}{n}}\right) e^{i\frac{2k\pi}{n}}$$

$$= c_0\, \omega_n^k$$

$\Rightarrow$ $z_0^{\frac{1}{n}} = \left\{ \text{"principal n-root of } z_0\text{"} \times \omega_n^k : k=0,1,\cdots,n-1 \ \& \ \omega_n = e^{i\frac{2\pi}{n}} \right\}$

[ω_n is called the n-root of unity]

§11 Examples

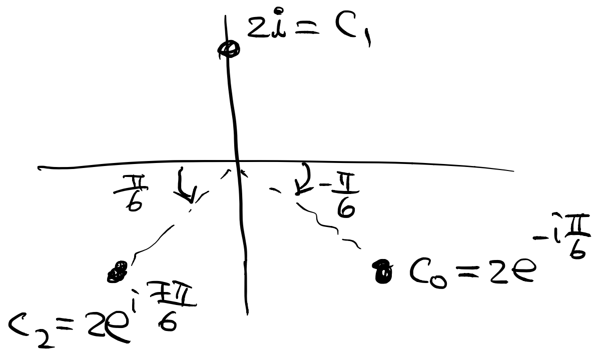
eg1 Find $(-8i)^{1/3}$

Soln: $-8i = 8e^{i(-\frac{\pi}{2})}$ $\quad \left(-\frac{\pi}{2} = \mathrm{Arg}(-8i)\right)$

$$\Rightarrow \quad (-8i)^{1/3} = \sqrt[3]{8}\, e^{i(-\frac{\pi}{6})}\, e^{i\frac{2k\pi}{3}}, \quad k = 0, 1, 2$$

$$= \left\{ 2e^{-\frac{i\pi}{6}},\ 2e^{i\frac{\pi}{2}},\ 2e^{i\frac{7\pi}{6}} \right\} \quad \text{(check!)}$$

$$= \left\{ \sqrt{3} - i,\ 2i,\ -\sqrt{3} - i \right\}$$

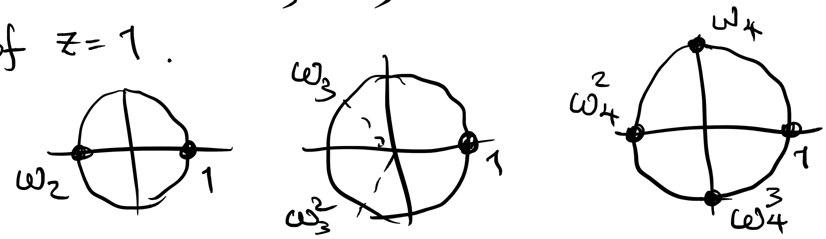


eg2 n-roots of unity

$$1^{\frac{1}{n}} = \sqrt[n]{1}\, e^{i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)} = e^{i\frac{2k\pi}{n}}$$

$$= \omega_n^k, \quad k = 0, 1, 2, \cdots, n-1.$$

i.e. $1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1}$ are all distinct n-roots of $z = 1$.



eg3 (Ex!) $(\sqrt{3}+i)^{1/2} = \pm \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right)$

§12 Regions in the complex plane

Def: (1) $B_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$ is called the ε-neighborhood (ε-nbd) of the point z_0

(2) $B_\varepsilon(z_0) \setminus \{z_0\} = \{ z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \}$ is called the deleted neighborhood. (or deleted ε-nbd)

Def: Let $S \subset \mathbb{C}$ be a subset.

(1) z_0 is said to be an interior point of S if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z_0) \subset S$,

& interior of S = set of interior points of S.

(2) z_0 is said to be an exterior point of S if $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(z_0) \subset \mathbb{C} \setminus S$.

(i.e. $B_\varepsilon(z_0) \cap S = \emptyset$)

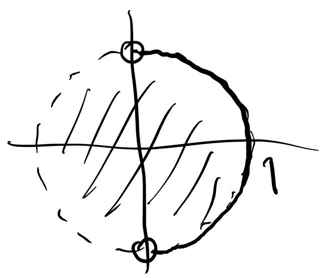
& exterior of S = set of exterior points of S.

(3) If z_0 is neither an interior point nor an exterior point, then it is called a <u>boundary point</u> of S,

& <u>boundary of S</u> = set of boundary points of S.

$\overset{\shortparallel}{(\partial S)}$

eg: $S = \{ z : |z| < 1 \text{ or } "|z| = 1 \text{ \& } \operatorname{Re} z > 0" \}$



interior of $S = \{|z|<1\}$

exterior of $S = \{|z|>1\}$ (Ex!)

boundary of $S = \{|z|=1\}$

<u>Def</u>: (1) A set $S \subset \mathbb{C}$ is called <u>open</u>

if $S \cap \partial S = \emptyset$ ($\Leftrightarrow$ S = interior of S)

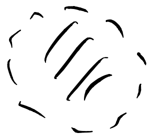
(2) A set $S \subset \mathbb{C}$ is called <u>closed</u>

if $\partial S \subset S$

(3) <u>closure of S</u> $\stackrel{\text{def}}{=\!=}$ $S \cup \partial S$.

(Note: S is closed $\Leftrightarrow$ S = closure of $S = S \cup \partial S$)

egs: (1)  is neither open nor closed

(2)  $\{|z|<1\}$ is open

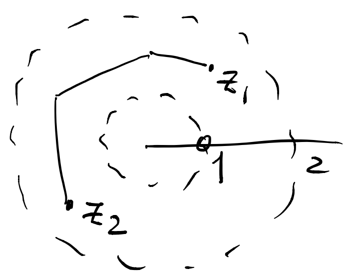
$\{|z|\le 1\}$ is closed.

<u>Def</u>: An open set S is <u>connected</u> if $\forall z_1, z_2 \in S$,

$\exists$ a <u>polygonal line</u> in S joining z_1 & z_2

(finite union of line segments joined end to end.)

<u>eg</u>: <u>Annulus</u> $\{1<|z|<2\}$ is <u>connected</u>:



<u>Def</u>: A <u>nonempty</u> <u>open</u> <u>connected</u> set is called a <u>domain</u>.

<u>eg</u>: $B_\varepsilon(z_0)$, $\{a<|z|<b\}$ are domains.
$(\varepsilon>0)$ $(0<a<b)$

<u>Def</u>: A set S is <u>bounded</u>, if $\exists R>0$ s.t.

$$S \subset B_R(0)$$

(i.e. $|z|<R,\ \forall z\in S$)

eg: $B_\varepsilon(z_0)$ ($\varepsilon>0$), $\{a<|z|<b\}$ ($0<a<b<+\infty$) are bounded

$\{\operatorname{Re} z>0\}$, $\{a<|z|<+\infty\}$ are <u>unbounded</u>

<u>Def</u>: A point z_0 is said to be an <u>accumulation point</u> of S, if

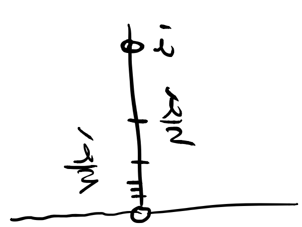
$$\forall \varepsilon>0, \ (B_\varepsilon(z_0)\setminus\{z_0\})\cap S \neq \emptyset$$

$$\left(\text{i.e. } \{0<|z-z_0|<\varepsilon\}\cap S\neq\emptyset, \ \forall\varepsilon>0\right)$$

<u>Facts</u>: (i) Any accumulation point of S belongs to the closure of S.

(ii) Thus a set S is closed $\Leftrightarrow$ S contains all of its accumulation pts.

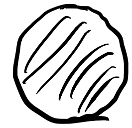
<u>egs</u>: (i) $S=\{i, \frac{i}{2}, \cdots \frac{i}{n}, \cdots\}$



$z=0$ is the only accumulation of S.

& $z=0\notin S$

$\Rightarrow$ S is not closed.

(ii)  $=S$, set of accumulation points of S = 
(check!)

Ch2 Analytic Functions

§13 Functions and Mappings

Let S be a set of cpx numbers

Def: (1) A function f defined on S is a rule that assigns to each $z \in S$, a complex number $w \underline{\underline{\text{denoted}}} f(z) \in \mathbb{C}$.

(2) The cpx number $w = f(z)$ is called the value of f at z.

(3) S is called the domain (of definition) of f

Convention: When the domain of f is not mentioned, we agree that the largest possible set is to be taken.

If $z = x + iy$ and $w = f(z) = u + iv$

i.e. $u + iv = f(z) = f(x+iy)$

$\Rightarrow$ u, v the real and imaginary parts of f are real-valued functions of 2-variables:

$$u = u(x,y) \quad \& \quad v = v(x,y)$$

& $\underline{f(z) = u(x,y) + i v(x,y)}$.

eg: $f(z) = z^2$, then

$$u + iv = f(z) = (x+iy)^2$$
$$= (x^2 - y^2) + 2ixy$$

$$\Rightarrow \quad \begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

Convention: If $f = u + iv$ with $v \equiv 0$, then f is a <u>real-valued</u> function of a <u>cpx variable</u>.

eg: $f(z) = |z|^2 = x^2 + y^2$

$$\begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$$

<u>Terminology</u>:

(1) $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_n \neq 0$ is a <u>polynomial of degree</u> n.

(2) Quotient $\frac{P(z)}{Q(z)}$ of polynomials $P(z)$ & $Q(z)$ are called <u>rational functions</u> (defined at z with $Q(z) \neq 0$)

Using polar coordinates or exponential forms of z:

$$\begin{cases} u = u(r,\theta) \\ v = v(r,\theta) \end{cases}$$

& we may write $\boxed{f(z) = u(r,\theta) + i\, v(r,\theta) \quad \text{for } z = re^{i\theta}}$

eg $w = z^2$ with $z = re^{i\theta}$

$$\Rightarrow \quad w = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2\cos 2\theta + i r^2 \sin 2\theta$$

$$\therefore \quad \begin{cases} u = r^2 \cos 2\theta \\ v = r^2 \sin 2\theta \end{cases}$$

Multiple-valued functions : assigns more than one value to a point z in the domain of definition.

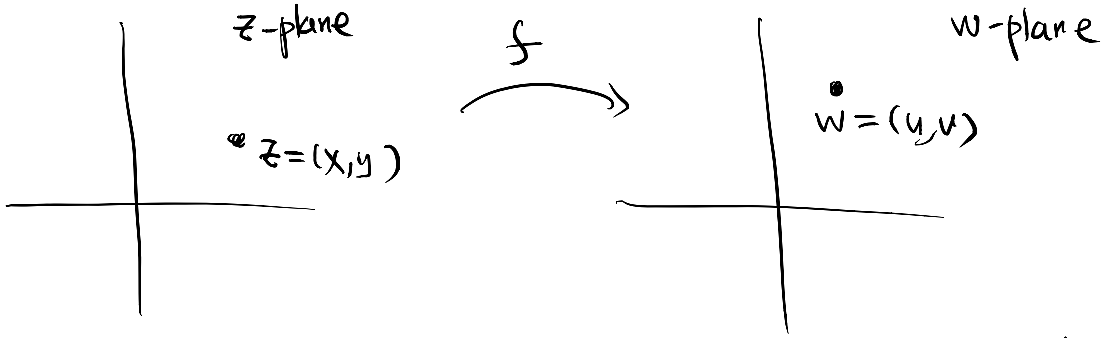
eg: $z \mapsto z^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$, $k = 0, 1, \cdots, n-1$

is a multiple-valued function for $n \geq 2$.

Terminology

(1) Mapping or transformation

when a function f is thought of correspondence between points $z = (x,y)$ & $w = (u,v)$:



(2) The point $w=(u,v)$ is called the <u>image</u> of the point $z=(x,y)$ under the mapping (transformation)

$$w=f(z).$$

(3) <u>Range of f</u> $= \{ w : w=f(z),\ \forall z \in S \}$

(4) <u>Inverse image</u> of a point w_0

$$f^{-1}(w_0) \stackrel{\text{def}}{=} \{ z \in S : f(z) = w_0 \}$$

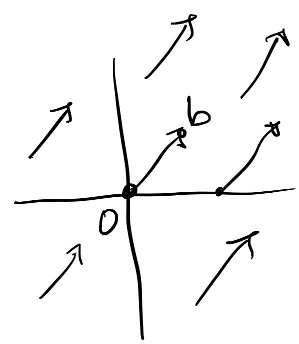
(Note: $f^{-1}(w_0)$ is in general a set of cpx numbers (or even empty) unless f is a one-to-one mapping.)

<u>Egs</u>: (1) For any fixed $b \in \mathbb{C}$

$$w = f(z) = z + b$$

is a <u>translation</u>

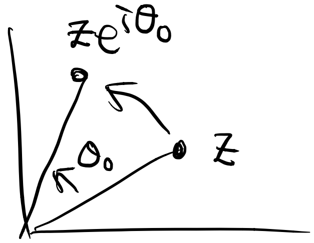
(when f is thought of a transformation)



(2) For any fixed $\theta_0 \in \mathbb{R}$

$$w = f(z) = e^{i\theta_0} z$$

is a rotation by angle θ_0 in counterclockwise direction



(3) The function $w = f(z) = \bar{z}$ is a reflection in x-axis