Ch 7 Hyperbolic Geometry

This is the non-Euclidean geometry discovered by Gauss, Bolyai, and Lobatchevsky.

There are 2 models of hyperbolic geometry to be discussed in this course:

<u>disk model</u> and

<u>upper half-plane model</u>.

<u>Remark</u>:
- Unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$
- Upper half-plane $\mathbb{U} = \{ z \in \mathbb{C} : z = x+iy, y > 0 \}$

<u>Disk model</u>:

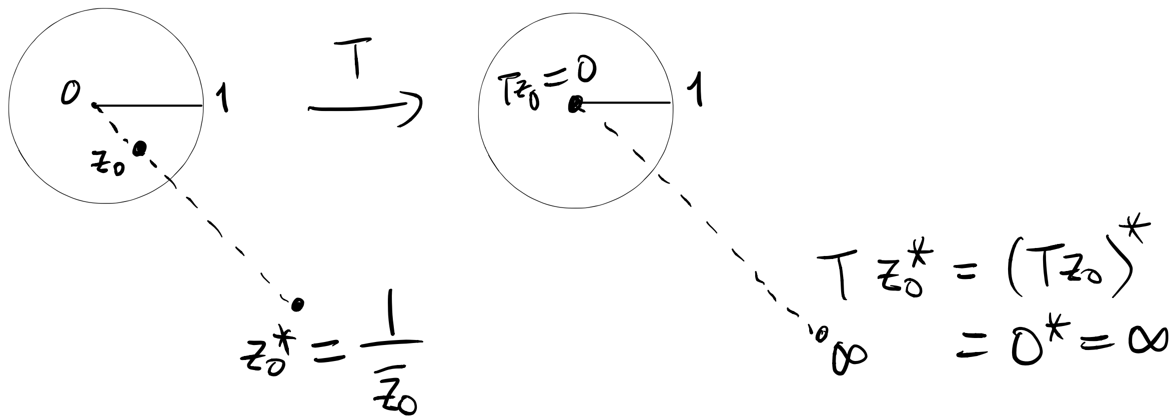
The group of transformations consists of all Möbius transformations that <u>map $\mathbb{D}$ onto itself</u>.

- It is clear that these transformations form a transformation group with underlying space $\mathbb{D}$. (Ex!)
- To find this group explicitly, we let $T \in$ Möb mapping $\mathbb{D}$ onto itself.

Then $|z|<1 \iff |Tz|<1$ $\left(\Rightarrow |z|=1 \iff |Tz|=1\right)$

And $\exists\ z_0 \in \mathbb{D}$ such that

$$Tz_0 = 0$$



Therefore, T mapping $\mathbb{D}$ onto itself

$$\Rightarrow \quad T(z_0^*) = \infty$$

i.e. $$T\left(\frac{1}{\bar{z}_0}\right) = \infty$$

Hence, we have

$$Tz = \alpha \frac{z - z_0}{z - \frac{1}{\bar{z}_0}} \quad \text{for some } \alpha \in \mathbb{C}\setminus\{0\}$$

$$= \left(-\alpha \bar{z}_0\right) \cdot \frac{z - z_0}{1 - \bar{z}_0 z}$$

$$= \lambda \frac{z - z_0}{1 - \bar{z}_0 z} \quad \text{where } \lambda = -\alpha \bar{z}_0$$

Suppose now $|z|=1$, then $|Tz|=1$

$$\therefore \quad 1=|Tz|=\left|\lambda \frac{z-z_0}{1-\bar{z}_0 z}\right|$$

$$=|\lambda| \frac{|z-z_0|}{|1-\bar{z}_0 z|}$$

$$=|\lambda| \frac{|z-z_0|}{|\bar{z} z-\bar{z}_0 z|}$$

$$=\frac{|\lambda|}{|z|} \cdot \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|}=|\lambda|$$

$\Rightarrow \quad \lambda=e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Hence we define

<u>Def</u>: Let $\mathbb{D}$ be the unit disk in the complex plane.

- Let $\mathbb{H}$ be the set of transformations of $\mathbb{D}$ of the form $Tz=e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}$, where $|z_0|<1$, $\theta \in \mathbb{R}$.
- The pair $(\mathbb{D}, \mathbb{H})$ <u>models hyperbolic geometry</u>.
- The set $\mathbb{D}$ will be called the <u>hyperbolic plane</u>.
- The group $\mathbb{H}$ is the <u>hyperbolic group</u>.

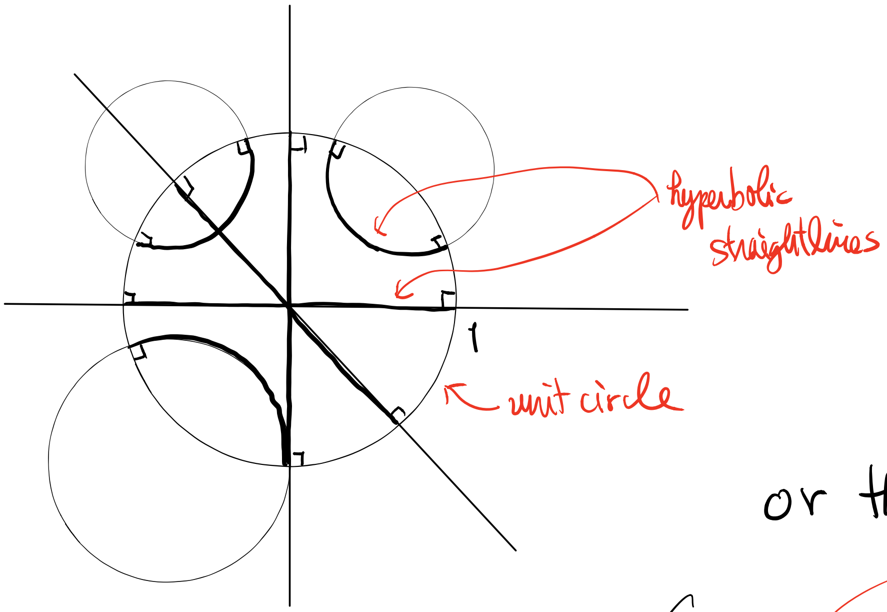
<u>Note</u>: $\mathbb{H}$ is a "subgroup" of the Möbius group $\mathbb{M}$ and $\mathbb{D} \subset \hat{\mathbb{C}}$.

$\Rightarrow$ hyperbolic geometry is a "<u>subgeometry</u>" of Möbius geometry.

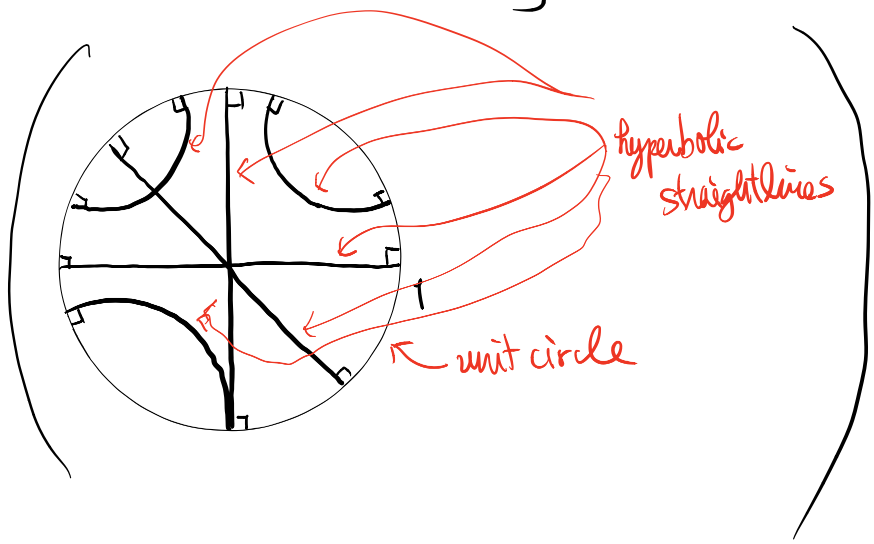
Hence: "Every" statement true in Möb geometry is also "true" in hyperbolic geometry!

<u>(Hyperbolic) Straight lines</u>

<u>Def</u> A (<u>hyperbolic</u>) <u>straight line</u> is (the part inside the unit disk) a Euclidean circle or Euclidean straight line in the complex plane that <u>intersects</u> the unit circle at a <u>right angle</u>.



or the following figure



Thm

(i) In hyperbolic geometry, all hyperbolic straight lines are congruent.

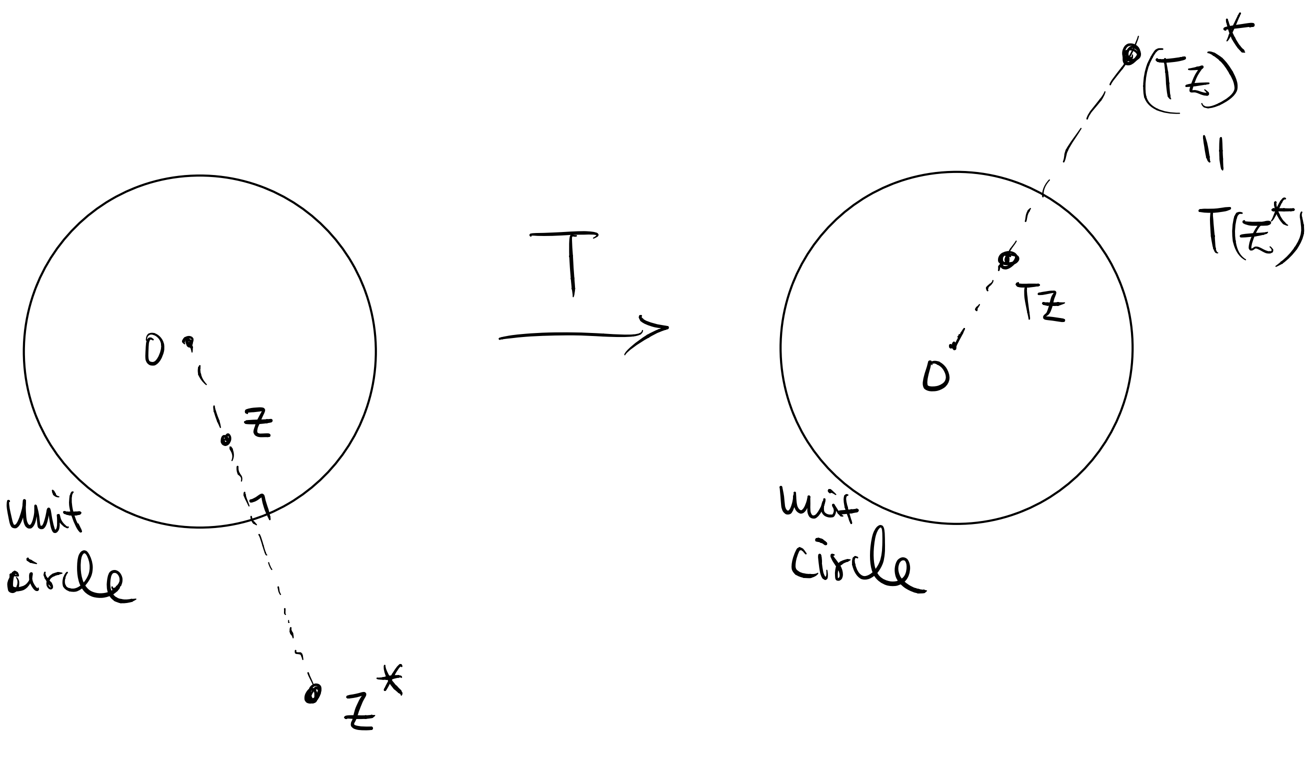
(ii) Two points in the hyperbolic geometry determine a <u>unique</u> hyperbolic straight line.

Recall: " $T(\underbrace{z^*}_{\text{wrt } \{|z|=1\}}) = \underbrace{(Tz)^*}_{\text{wrt } \{|z|=1\}}$ " in our case of

(T = transformation in hyperbolic group)

of unit circle can be written as

$$\boxed{\begin{array}{l}\text{Lemma 1: Each transformation of hyperbolic geometry}\\ \text{maps each pair of points symmetric wrt the}\\ \text{unit circle to another pair of points symmetric}\\ \text{wrt the unit circle.}\end{array}}$$

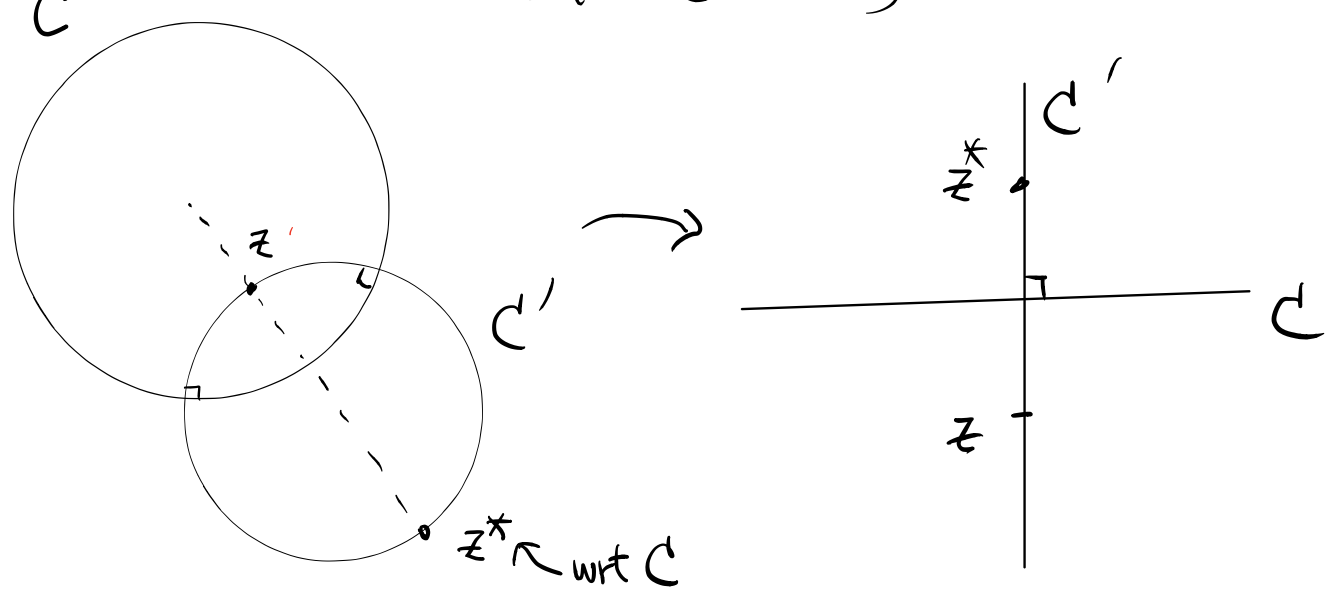


We also need the following:

Lemma 2: Let C be a cline. Let z & z^* be distinct symmetric points wrt C. Then any cline C' that is athogonal to C and passing throught z must also passing throught z^*.

Conversely, any cline that passes throught z & z^* is orthogonal to C.

Pf of Lemma 2: Transfoms the figure to the figure on the RHS,

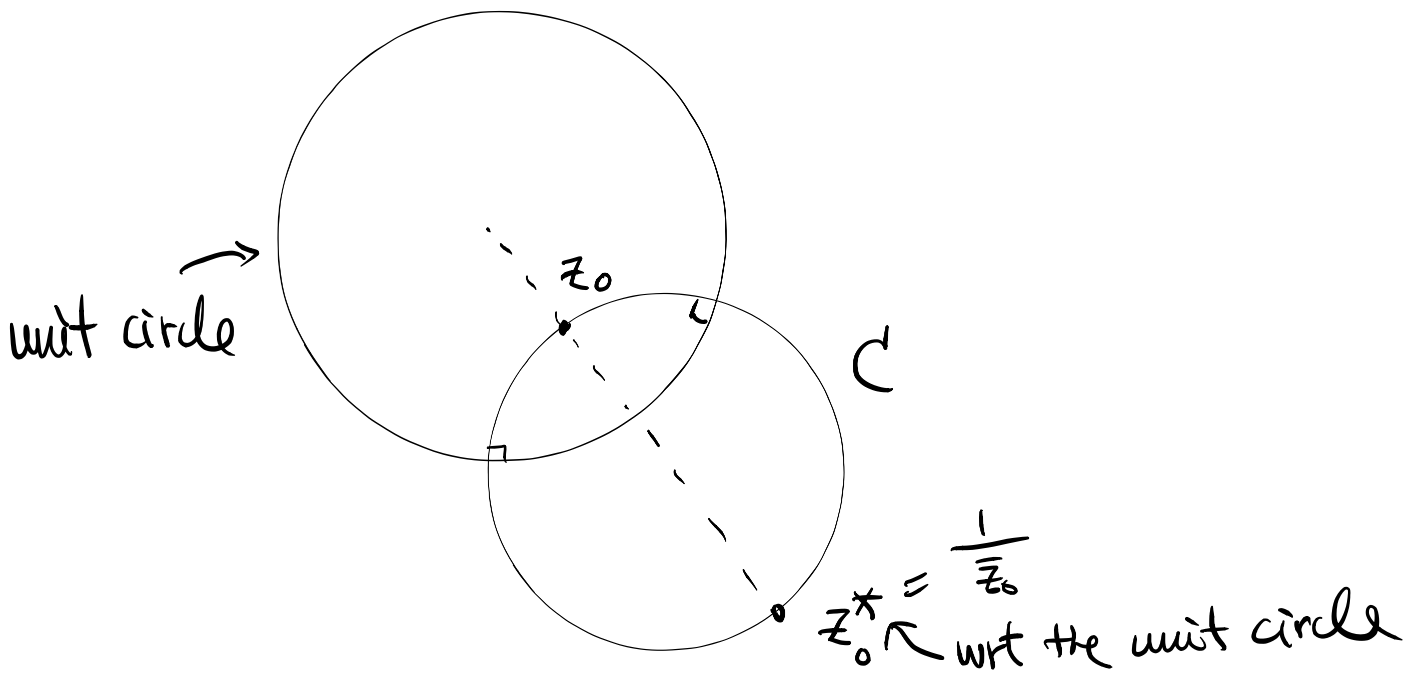


then use Question 5 of HW1. #

Proof of the Thm :

(i) Let C be a cline that is athogonal to the unit circle, i.e. C is a hyperbolic straight line

And let z_0 be a point on C.



By Lemma 2, z_0^* also lies on C (outsides $\mathbb{D}$), where $*$ = sym. wrt the unit circle.

Let $Tz = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ (θ chosen later)

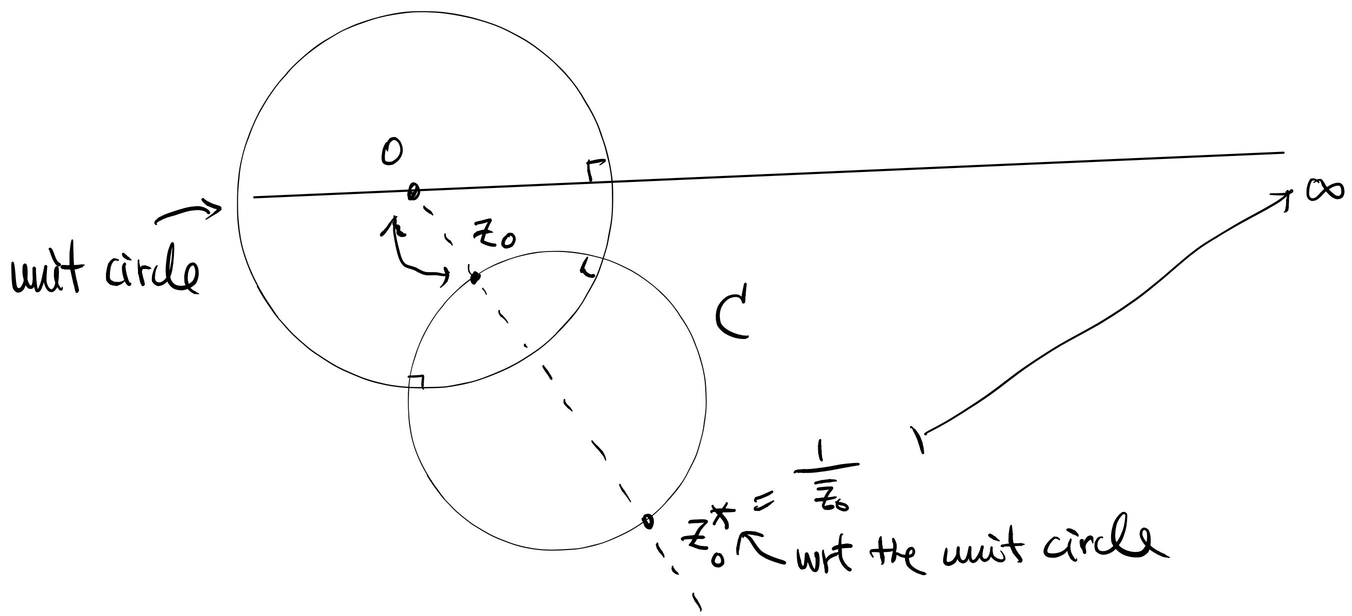
Then T is a transformation of hyperbolic geometry (i.e. $T \in \mathbb{H}$ the hyperbolic group) and

$$Tz_0 = 0 \,, \quad T(z_0^*) = T\left(\frac{1}{\bar{z}_0}\right) = \infty$$

$\therefore$ $T(C)$ is a cline passing throught 0 & ∞, and orthogonal to $\{|z|=1\}$.

$\Rightarrow$ $T(C)$ must be a diameter of the unit circle

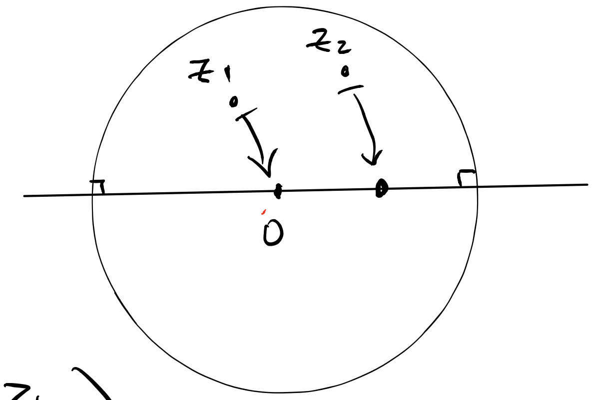
Finally, we can choose θ so that $T(C) = x$-axis. (Ex!)



This proves that any hyperbolic straight line C is congruent to the x-axis. And hence all hyperbolic straight lines are congruent. This proves part (i).

For part (ii), let z_1, z_2 be any 2 distinct points in $\mathbb{D}$. Then

$$Tz = e^{i\theta}\frac{z - z_1}{1 - \bar{z}_1 z}$$



takes z_1 to 0.

Choose $\theta = -\arg\left(\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right)$

Then
$$Tz_2 = e^{i\theta}\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}$$
$$= e^{i\theta}\left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| e^{i\arg\left(\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right)}$$
$$= \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| > 0$$
(positive real number, and in fact < 1.)

Note that x-axis is the unique hyperbolic straight line passing through 0 and Tz_2. This proves that T^{-1}(x-axis) is the unique hyperbolic straight line passing through z_1 & z_2.

#

Euclid's Postulates

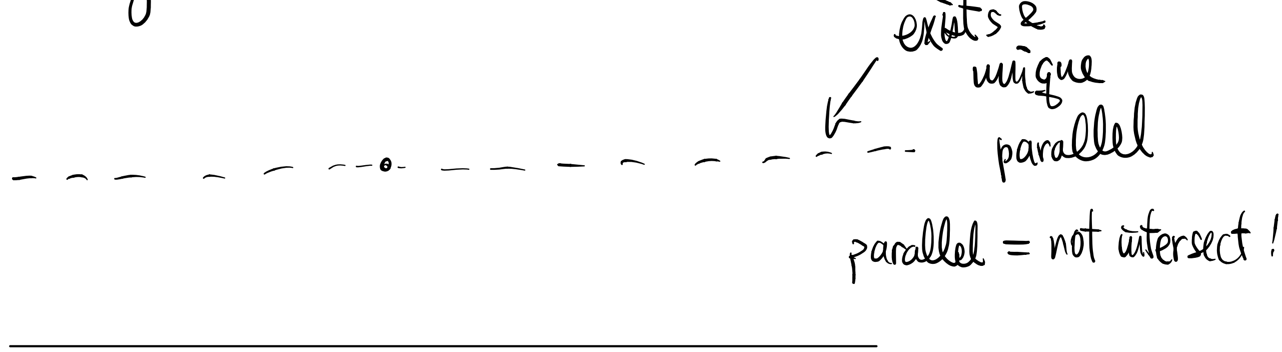
Postulate 1 : Two points determine a straight line.

Postulate 2 : A line can be produced indefinitely in either direction.

Postulate 3 : A circle can be described with any center and radius.

Postulate 4 : All right angles are congruent.

Postulate 5 : Through a point not on a line, there is a unique line parallel to the given line.

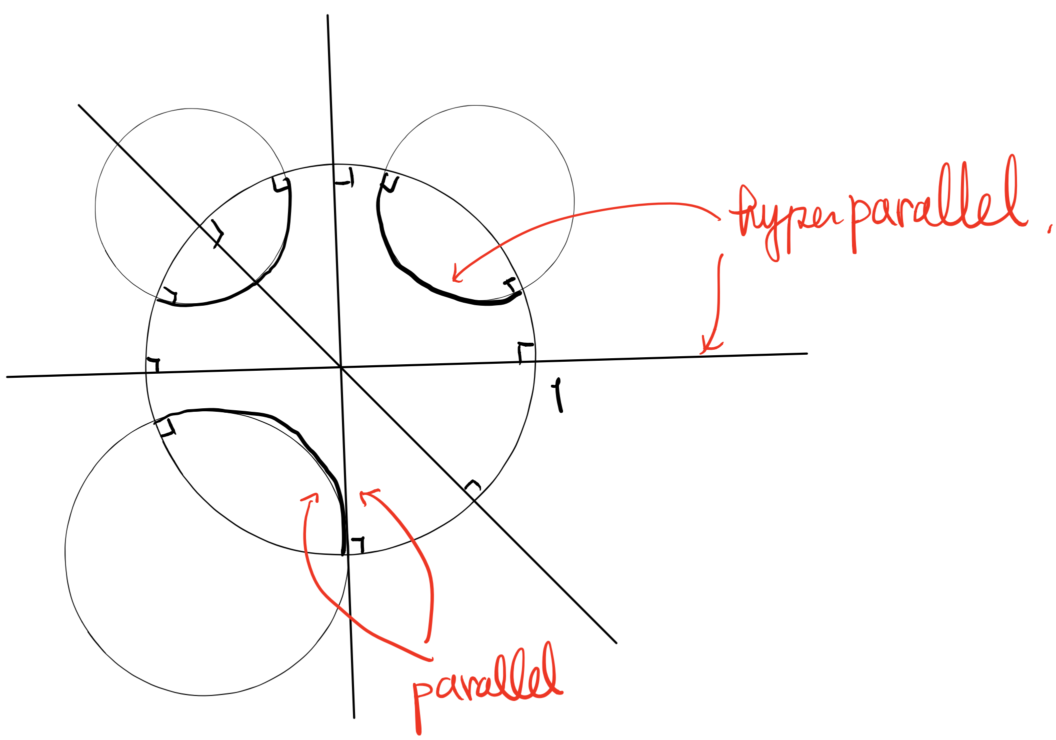


Parallelism in hyperbolic geometry

Def: (i) The points on the unit circle are called <u>ideal points</u>.

(ii) Two hyperbolic lines are called <u>parallel</u> if they do not intersect inside $\mathbb{D}$ but do <u>share one ideal point</u>.

(iii) Two hyperbolic lines are called <u>hyperparallel</u> if they do not intersect inside $\mathbb{D}$ and do <u>not have</u> an ideal point in common.

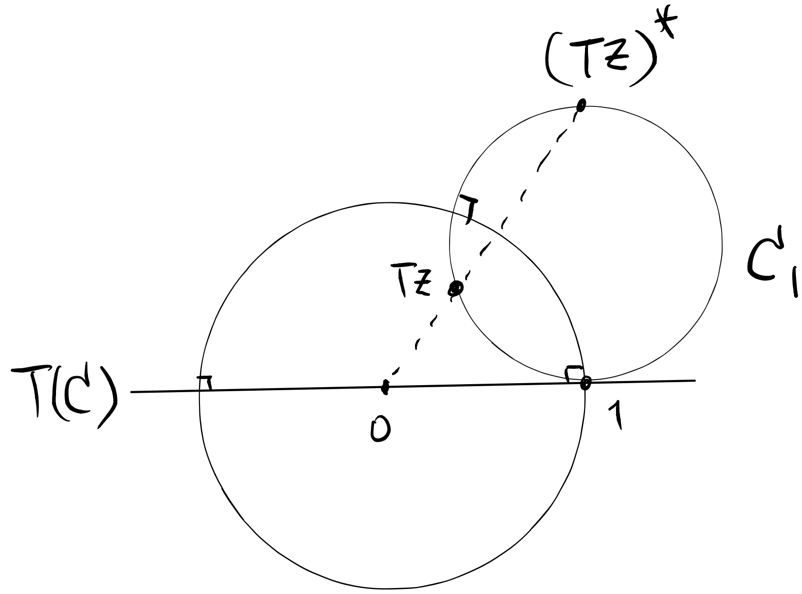


Postulate 5 is false in hyperbolic geometry.

In fact, $\forall$ point p not on a hyperbolic line C, there exists 2 hyperbolic lines parallel to C and passing thro. p.

Pf: For any hyperbolic line C, $\exists$ transformation $T \in \mathbb{H}$ such that $T(C) = x$-axis.

If $z \in \mathbb{D}$ is a point not on C, then Tz is a point not on $T(C) = x$-axis



Let C_1 be the unique Euclidean circle passing thro. the points 1, Tz, and $(Tz)^*$. (since Tz not on $T(C) =$ x-axis, $Tz \neq 0$ & hence $(Tz)^* \neq \infty$)

Lemma 2 $\Rightarrow$ C_1 is orthogonal to the unit circle.

$\Rightarrow$ C_1 & $T(C) =$ x-axis tangent at 1, and hence they have no other intersection.

$\Rightarrow$ C_1 is parallel to $T(C)$ and passing thro. Tz.

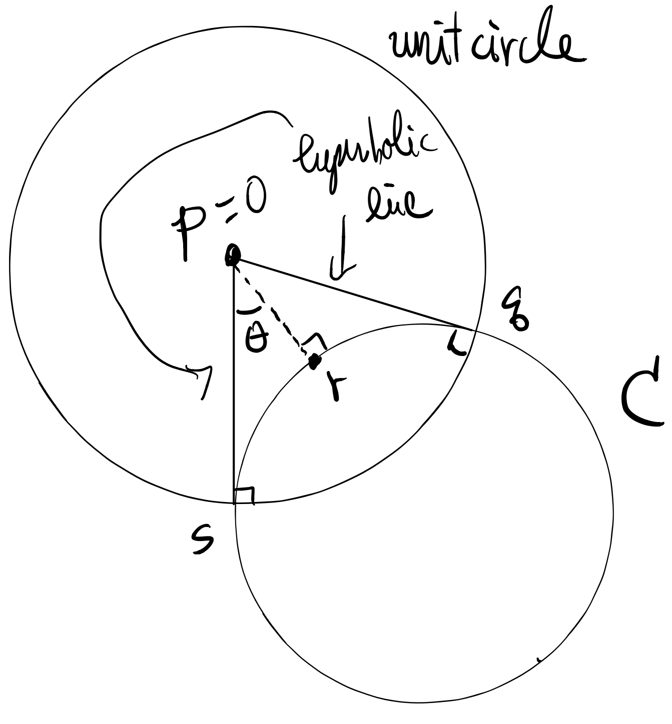
$\Rightarrow$ $T^{-1}(C_1)$ is parallel to C and passing thro. z.

Similarly, we can find C_{-1} passing thro -1, Tz, $(Tz)^*$. and $T^{-1}(C_{-1})$ is parallel to C and passing thro. z.

Since $C_1 \neq C_{-1}$, Postulate 5 is false. ※

Angle of parallelism

Def. Let $C = \overline{srq}$ be a hyperbolic straight line and $p \in \mathbb{D}$ be a point not on C such that the hyperbolic straight line passing thro p & r is perpendicular to C. Then the angle θ between one of the parallels ($\overline{ps}$ or $\overline{pq}$) and the perpendicular $\overline{pr}$ is called the <u>angle of parallelism</u>.



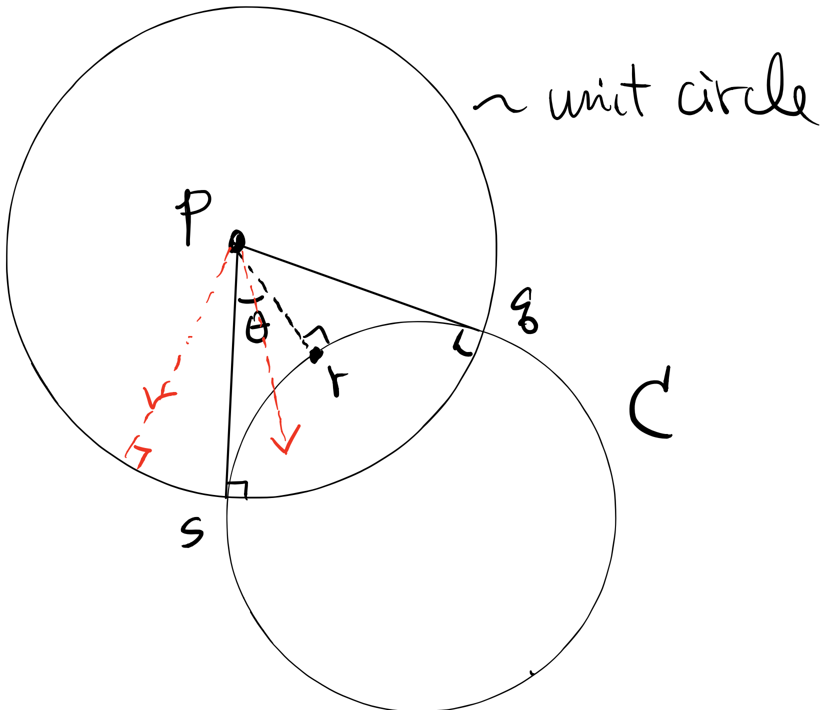
(After a suitable hyperbolic transformation, we may assume p = origin (ie $z=0$))

Remark (Ex!): A ray passing thro. p makes

(i) an angle with $\overline{pr} < \theta$, then it intersects $\overline{srq}$.

(ii) an angle with $\overline{pr} = \theta$, then it is parallel to $\overline{srq}$.

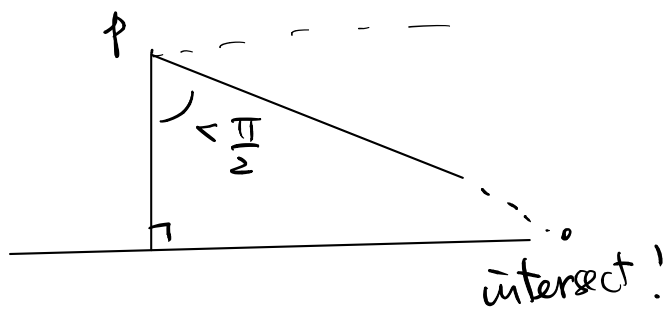
(iii) an angle with $\overline{pr} > \theta$, then it is hyperparallel to $\overline{srq}$.



Remark: The angle of parallelism is always acute.

(Easy Ex: transform p to 0)

[**Compare**: Euclidean geometry

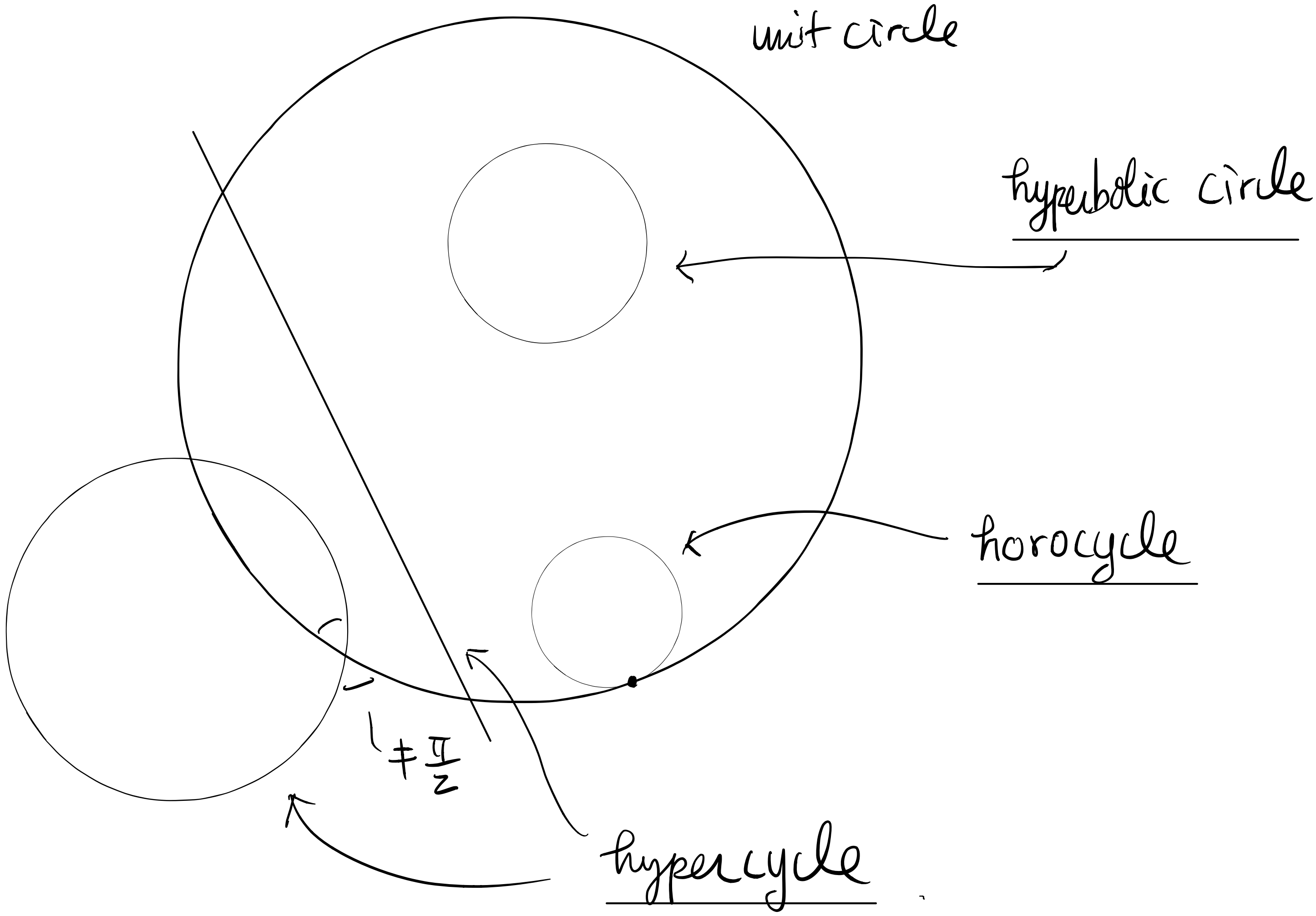


]

Ch8 Cycles

<u>Def</u>: Let C be a portion of a Euclidean circle or straight line inside the unit disk. Suppose that C is <u>not</u> perpendicular to the unit circle. Then C is called a <u>cycle</u>.

- If C is entirely contained in $\mathbb{D}$, then C is a <u>hyperbolic circle</u>.
- If C is <u>tangent</u> to the unit circle, then C is a <u>horocycle</u>.
- If C <u>intersects</u> the unit circle (at an angle $\neq \frac{\pi}{2}$) C is a <u>hypercycle</u>.


unit circle
hyperbolic circle
horocycle
hypercycle

By Lemma 1, if T is a transformation of the hyperbolic group, then T maps $\mathbb{D}$ onto itself and hence $T(\partial\mathbb{D}) = \partial\mathbb{D}$. If T has a fixed point inside $\mathbb{D}$, then the symmetric point (wrt $\partial\mathbb{D}$) outside $\mathbb{D}$ is also a fixed point of T: $T(z^*) = (Tz)^* = (z)^*$.

So we can analyze $T \in \mathbb{H}$ ($T \neq Id$) by the following situations (using cycles)

(A) 1 fixed point inside $\mathbb{D}$ & 1 fixed point outside $\mathbb{D}$

(B) 2 fixed points on $\{|z|=1\} = \partial\mathbb{D}$.

(C) 1 fixed point only (must be on $\{|z|=1\} = \partial\mathbb{D}$)