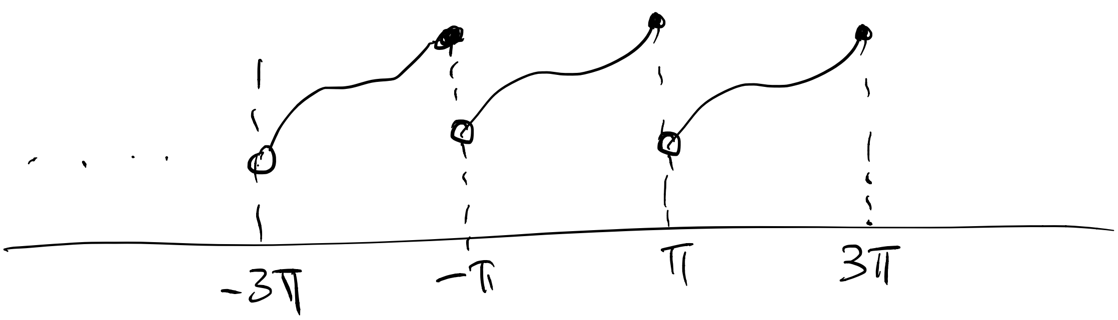
Note: For any Riemann integrable function f on $[-\pi, \pi]$, we can define all the $a_0, a_n, b_n, n \geq 1$ as in the defn., & hence a Fourier series.

On the other hand, we can restrict a f to $(-\pi, \pi]$ and extend periodically to a 2π-periodic function $\tilde{f}$ on $\mathbb{R}$:



And according to the defn. of Fourier coefficients, f & $\tilde{f}$ have the same Fourier Series!

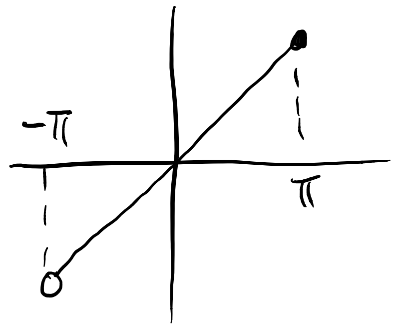
So we will not distinguish f & $\tilde{f}$!

Notation We use

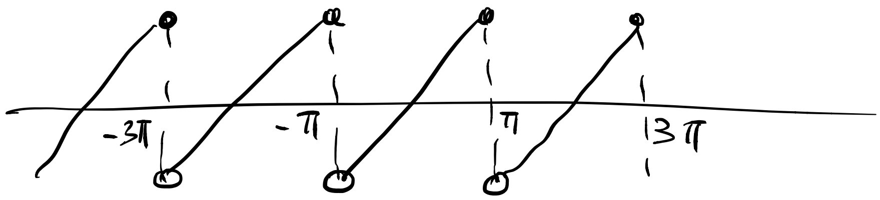
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

to denote if the trigonometric series on the RHS is the Fourier Series of f.

(does not indicate the series converges to f in any sense) eg 1.1 $\quad f_1(x) = x$ restricted to $(-\pi, \pi]$



Extension to 2π-periodic function $\tilde{f}_1$ on $\mathbb{R}$



$\tilde{f}_1(-\pi) = \pi \neq -\pi$

$\tilde{f}_1$ = odd function

$$
\begin{cases}
a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} x\,dx = 0 \\
a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\cos nx\,dx = 0 \\
b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\sin nx\,dx = (-1)^{n+1}\dfrac{2}{n} \quad (\text{Check!})
\end{cases}
$$

$$\therefore \quad f_1(x) = x \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx$$

$(\sim \tilde{f}_1(x))$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$ is a sine series ($\because f_1$ is odd)

Notes: (1) For $x = \pm\pi$, Fourier series$\Big|_{\pm\pi} = 0$

But $f_1(\pm\pi) = \pm\pi$

$\tilde{f}_1(\pm\pi) = \pi$ $\Big\} \neq$ Fourier series$\Big|_{\pm\pi}$

(2) Convergence is not clear (for $x \neq \pm\pi$)

as the terms decay like $\frac{1}{n}$ & $\sum \frac{1}{n}$ doesn't converge.

<u>Notation</u>: "Big O" & "small o"

Let $\{x_n\}$ be a sequence, then

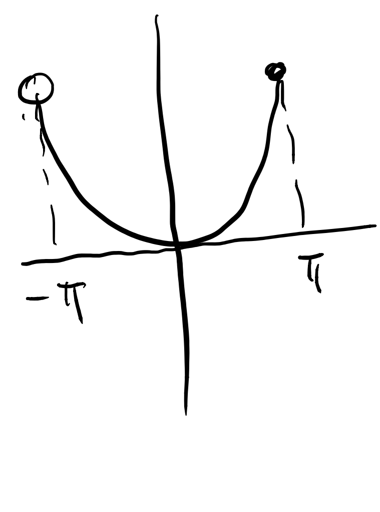
(i) $x_n = O(n^s) \Leftrightarrow |x_n| \leq Cn^s$ for some const. $C > 0$ (as $n \to \infty$)

(ii) $x_n = o(n^s) \Leftrightarrow \frac{|x_n|}{n^s} \to 0$ as $n \to \infty$

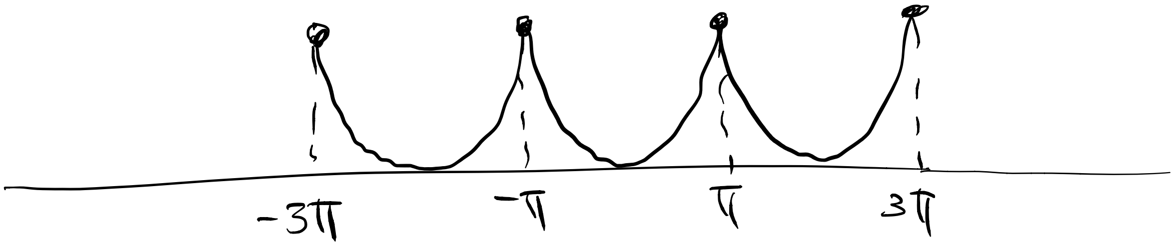
(egs): (i) $x_n = \frac{2(-1)^{n+1}}{n} \sin nx = O(\frac{1}{n})$ $\left(|x_n| \leq \frac{2}{n} \right)$

(ii) $x_n = \log n = o(n)$ $\left(\frac{\log n}{n} \to 0 \text{ as } n \to \infty \right)$

<u>Eg 1.2</u> $f_2(x) = x^2$ restricted to $(-\pi, \pi]$



Extension to a 2π-periodic function $\tilde{f}_2$ on $\mathbb{R}$



$\tilde{f}_2$ is continuous (since $f_2(-\pi) = f_2(\pi)$)

$\tilde{f}_2$ is an even function

It is an easy exercise of integration to find that

$$f_2(x) \equiv x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \quad (\text{Ex!})$$

One sees that $\qquad a_n = O\left(\frac{1}{n^2}\right) \Rightarrow \sum |a_n| < \infty$

$\Rightarrow$ Fourier series converges uniformly to a continuous function.

(Will it be the function $\tilde{f}_2$? See later discussion)

Observation: Egs 1 & 2 : $\left\{ \begin{array}{l} \text{odd function} \longrightarrow \text{Sine series} \\ \text{even function} \longrightarrow \text{Cosine series} \end{array} \right.$

This is true in general ! (Ex !)

Complex Fourier Series

Def: (1) A complex trigonometric Series is a series of the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$.

($\{c_n\}_{-\infty}^{\infty}$ is called a bisequence of cpx members & $\{c_n e^{inx}\}_{n=-\infty}^{\infty}$ is a bisequence of cpx-valued functions)

(2) $\sum_{-\infty}^{\infty} c_n e^{inx}$ is said to be convergent at x if $\lim_{N \to +\infty} \sum_{n=-N}^{N} c_n e^{inx}$ exists.

Def: Complex Fourier Series of a 2π-periodic cpx-valued function f which is integrable on $[-\pi, \pi]$, denoted by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is a cpx trigonometric series with (complex) Fourier coefficients c_n defined by

$$\boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \quad \forall n \in \mathbb{Z}}$$

Notes (i) For cpx-valued function $f = u + iv$ with u, v real-valued

$$\int_a^b f \stackrel{\text{def}}{=} \left(\int_a^b u\right) + i\left(\int_a^b v\right)$$

(ii) f is called integrable $\Leftrightarrow$ both u & v are integrable.

Motivation for cpx Fourier Series:

"If" $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ & "converges nicely"

Then $$f(x) e^{-imx} = \sum_{-\infty}^{\infty} c_n e^{inx} e^{-imx}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

It is easy to find

$$\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 2\pi, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Hence $$\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = c_m \cdot 2\pi$$

$$\Rightarrow \quad c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx$$

Relationship between (Real) Fourier Series & complex Fourier Series for a real-valued function f.

By $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx$$

$$= \frac{1}{2} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \right) - \frac{1}{2} i \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right)$$

Therefore:

for $n=0$, $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = a_0$

for $n \geq 1$, $c_n = \frac{a_n}{2} - i \frac{b_n}{2}$

for $n \leq -1$, then $(-n) \geq 1$ &

$$c_n = \frac{1}{2} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(-n) x \, dx \right) + \frac{1}{2} i \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(-n) x \, dx \right)$$

$$= \frac{1}{2} a_{(-n)} + i \frac{1}{2} b_{(-n)}$$

$\therefore$

$$\boxed{c_n = \begin{cases} \frac{1}{2}(a_n - i b_n) & \text{for } n \geq 1 \\ a_0 & \text{for } n = 0 \\ \frac{1}{2}(a_{(-n)} + i b_{(-n)}) & \text{for } n \leq -1 \end{cases}}$$

for real valued function.

Corollary: If f is a real-valued function, then

$$C_{-n} = \overline{C_n} \quad \xleftarrow{} \text{ cpx conjugate}$$

$\forall n \in \mathbb{Z}$.

(i.e. $C_n = \overline{C_{-n}}$)

Pf: $n \geq 1 \Rightarrow (-n) \leq -1$

$$\therefore \quad C_{-n} = \frac{1}{2}\left(a_{-(-n)} + i\, b_{-(-n)}\right)$$
$$= \frac{1}{2}(a_n + i b_n) = \overline{C_n}$$

Similar for others. $\divideontimes$

Prop: Let f be a 2π-periodic real function which is differentiable on $[-\pi, \pi]$ with f' integrable on $[-\pi, \pi]$. Denote the Fourier coefficients of f & f' by $\{a_n(f), b_n(f); C_n(f)\}$ & $\{a_n(f'), b_n(f'); C_n(f')\}$ respectively.

Then
$$\begin{cases} a_n(f') = n\, b_n(f) \\ b_n(f') = -n\, a_n(f) \end{cases}$$

& $$C_n(f') = in\, C_n(f)$$

(So it is more convenient to work with cpx Fourier coefficients when derivatives involved.)

Pf: $a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$

(integration by-parts) $= \frac{1}{\pi}\left[f(x)\cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n\sin nx)\,dx\right]$

$(f(\pi)=f(-\pi))$ $= \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = n\, b_n(f)$

Similarly for $b_n(f') = -n\, a_n(f)$ (Check)

For $c_n(f')$, either from the above formula relating c_n to a_n & b_n, or integration by parts directly

$$c_n(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx = \frac{1}{2\pi}\left[\cancelto{0}{f(x)e^{-inx}\Big|_{-\pi}^{\pi}} + (in)\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right]$$

$$= in\, c_n(f).$$

<u>Fourier Series of $2T$-periodic (real) functions</u>

Let f be a $2T$-periodic function

Then $g(x) = f(\frac{T}{\pi}x)$ is 2π-periodic

Therefore

$$f\left(\frac{T}{\pi}x\right) = g(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with
$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx & = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\, dx & = \frac{1}{T}\int_{-T}^{T} f(y)\cos\left(\frac{n\pi}{T}y\right)dy \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\, dx & = \frac{1}{T}\int_{-T}^{T} f(y)\sin\left(\frac{n\pi}{T}y\right)dy. \end{cases}$$

Hence

$$f(y) \sim a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{T}y\right) + b_n \sin\left(\frac{n\pi}{T}y\right)\right]$$

with
$$\begin{cases} a_0 = \frac{1}{2T}\int_{-T}^{T} f(y)dy \\ a_n = \frac{1}{T}\int_{-T}^{T} f(y)\cos\left(\frac{n\pi}{T}y\right)dy & (n \geq 1) \\ b_n = \frac{1}{T}\int_{-T}^{T} f(y)\sin\left(\frac{n\pi}{T}y\right)dy. \end{cases}$$

is called <u>Fourier series</u> of the <u>$2T$-periodic function f</u>.

§1.2 Riemann–Lebesgue Lemma

Recall: A step function on $[-\pi,\pi]$ is a function of the form

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$$

where (i) $I_j = (a_j, a_{j+1}]$ for $j=1,\cdots,N-1$

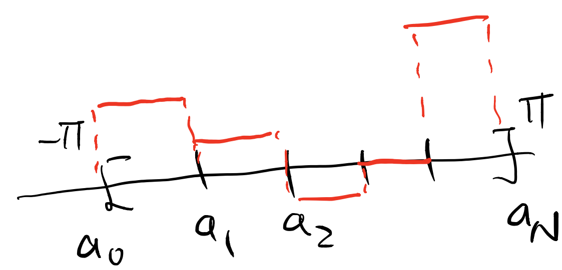
$I_0 = [a_0, a_1]$ with

$$-\pi = a_0 < a_1 < \cdots < a_{N-1} < a_N = \pi .$$

(ii) For a set E, $\chi_E = \begin{cases} 1, & x\in E \\ 0, & x\notin E \end{cases}$

is the characteristic function for E.

(iii) $s_j \in \mathbb{R}$, $j=0,\cdots,N-1$



Lemma 1.2 For every step function s integrable on $[-\pi,\pi]$, $\exists$ constant $C>0$ (indep. of n, but depends on s) such that

$$|a_n(s)|,\ |b_n(s)| \le \frac{C}{n}, \quad \forall n\ge 1$$

where $a_n(s)$, $b_n(s)$ are Fourier coefficients of s.

Pf: Let $s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}(x)$

We have, for $n \geq 1$

$$\pi a_n(s) = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{N-1} s_j \chi_{I_j}(x) \right) \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \frac{[\sin(n a_{j+1}) - \sin(n a_j)]}{n}$$

$$\Rightarrow \quad |a_n(s)| \leq \frac{C}{n}$$

where $\quad C = \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j| > 0$ (indep on n)

Similarly for $|b_n(s)| \leq \frac{C}{n}$ for all $n \geq 1$. ※

Lemma 1.3

Let f be integrable on $[-\pi, \pi]$. Then $\forall \varepsilon > 0$,

$\exists$ a step function $s(x)$ such that

(i) $\quad s \leq f$ on $[-\pi, \pi]$, &

(ii) $\quad \int_{-\pi}^{\pi} (f - s) < \varepsilon$

Pf: f (Riemann) integrable

$\Rightarrow$ f can be approximated from below by

Darbour lower sums

i.e. $\forall \varepsilon > 0$, $\exists$ partition $a_0 = -\pi < a_1 \cdots < a_N = \pi$

s.t.
$$\int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) < \varepsilon$$

where $m_j = \inf\{f(x) : x \in [a_j, a_{j+1}]\}$.

Define the step function
$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j} \qquad (\text{ie. } s_j = m_j)$$

with $I_j = (a_j, a_{j+1}]$ for $j = 1, \cdots, N-1$

$I_0 = [a_0, a_1]$

Then $s \le f$ & $\int_{-\pi}^{\pi} s(x)dx = \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j)$

$$\Rightarrow \int_{-\pi}^{\pi} (f - s) < \varepsilon. \quad \cancel{\times}$$

Now we can prove

Thm 1.1 (Riemann-Lebesgue Lemma)
The Fourier coefficients of a 2π-periodic function f integrable on $[-\pi, \pi]$ converge to 0 as $n \to +\infty$.

Pf: $\forall \varepsilon > 0$, Lemma 1.3 $\Rightarrow$ $\exists$ step function s s.t.
$$s \le f \quad \& \quad \int_{-\pi}^{\pi} (f - s) < \frac{\varepsilon}{2}.$$

& by Lemma 1.2, $\exists n_0 > 0$ s.t.

$$|a_n(s)| < \frac{\varepsilon}{2} \quad \forall n \geq n_0$$

(for instance, take $n_0 = \left[\frac{2C}{\varepsilon}\right] + 1$, where C is the constant given by Lemma 1.2)

Therefore $$|a_n(f) - a_n(s)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} (f-s)(x) \cos nx \, dx\right|$$
$$\leq \frac{1}{\pi}\int_{-\pi}^{\pi} f-s \quad (\text{as } f \geq s)$$
$$< \frac{\varepsilon}{2\pi}$$

Hence $$|a_n(f)| \leq |a_n(s)| + |a_n(f) - a_n(s)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\pi} < \varepsilon, \quad \forall n \geq n_0$$

$\therefore \quad a_n(f) \to 0$ as $n \to +\infty$.

Similarly for $b_n(f)$. ※

§1.3 Convergence of Fourier Series

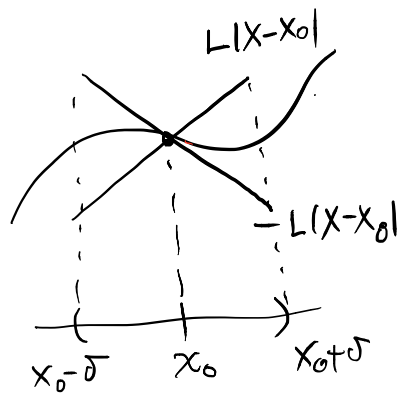
Terminology: For $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

we denote $$(S_n f)(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

the n-th partial sum of the Fourier series of f.

Def: Let f be a function on $[a,b]$. Then f is called Lipschitz continuous at $x_0 \in [a,b]$ if $\exists\, L>0$ & $\delta>0$ such that

$$|f(x)-f(x_0)| \le L|x-x_0|, \quad \forall\, |x-x_0|<\delta$$
$$(x \in [a,b])$$



Note (1) Both L & δ may depend on x_0.

Note: (2) If f is Lipschitz continuous at $x_0 \in [a,b]$ & f is bounded on $[a,b]$, then $\exists\, L'>0$ (L' may depends on x_0) s.t.

$$|f(x)-f(x_0)| \le L'|x-x_0|, \quad \forall\, x \in [a,b].$$

Pf: By defn. f Lip. cts at x_0

$\Rightarrow \exists\, L>0, \delta>0$ s.t.

$$|f(x)-f(x_0)| \le L|x-x_0|, \quad \forall\, |x-x_0|<\delta.$$

If $|x-x_0| \ge \delta$, then $\dfrac{|x-x_0|}{\delta} \ge 1$

$$\Rightarrow |f(x)-f(x_0)| \le |f(x)|+|f(x_0)| \le 2M$$
$$\le \frac{2M}{\delta}|x-x_0|$$

where $M = \sup\limits_{[a,b]}|f| > 0$.

Hence

$$|f(x)-f(x_0)| \le \begin{cases} L|x-x_0| , & |x-x_0|<\delta \\ \frac{2M}{\delta}|x-x_0| , & |x-x_0|\ge\delta \end{cases}$$

$$\Rightarrow \quad |f(x)-f(x_0)| \le L'|x-x_0| , \quad \forall x\in[a,b],$$

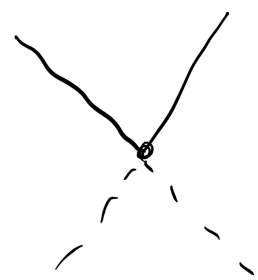
with $L' = \max\{L, \frac{2M}{\delta}\} > 0$. #

eg: $f\in C^1[a,b]$ (continuously differentiable on $[a,b]$)

$\Rightarrow$ f is Lipschitz cts. at every $x_0\in[a,b]$.

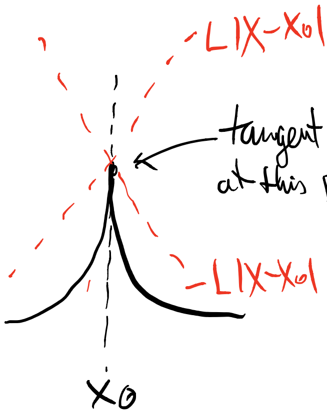
On the other hand $f(x)=|x|$ is Lip cts. at $x=0$, but not differentiable.

(Ex!)



(take $L=1$)

eg:



this graph gives a cts function at x_0, but not Lip. cts at x_0.

more precisely $f(x)=|x|^\alpha$ with $0<\alpha<1$ is not Lip cts. at $x=0$.