MATH2060 Solution 10

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9.1 Q7

(a) Since (b_n) is bounded, there exists some $K > 0$ such that $|b_n| \leq K$ for all n. Define $S_n = \sum_{k=1}^{n} |a_n b_n|$. Note that since S_n is an increasing sequence, to show that it is convergent, it suffices to show that S_n is bounded. Note that

$$S_n \leq K \sum_{k=1}^{n} |a_k|$$

The RHS is bounded since $\sum a_n$ is absolutely convergent. We conclude that $\sum a_n b_n$ is absolutely convergent.
(b) Let $a_n = (-1)^n \frac{1}{n}$ and $b_n = (-1)^n$. Note that $|b_n| \leq 1$, so it is bounded. Moreover, $\sum a_n$ is conditionally convergent. However, $\sum a_n b_n = \sum \frac{1}{n}$, which is divergent.

9.2 Q4

(a) Let $a_n = 2^n e^{-n} = (\frac{2}{e})^n$. Consider the ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{e} < 1$$

for all n. So the series $\sum a_n$ is convergent.
(f) Let $b_n = n! e^{-n^2}$. Consider the ratio test:

$$\left| \frac{b_{n+1}}{b_n} \right| = (n+1)e^{-(2n+1)}$$

Consider the function $f(x) = (x+1)e^{-(2x+1)}$ for $x \geq 1$. Note that $f'(x) = e^{-(2x+1)} - 2(x+1)e^{-(2x+1)} < 0$ for all $x \geq 1$. Hence, f is decreasing and we have $f(x) \leq f(1) = 2e^{-3} < 1$ for all $x \geq 1$. We conclude that

$$\left| \frac{b_{n+1}}{b_n} \right| = (n+1)e^{-(2n+1)} = f(n) \leq 2e^{-3} < 1$$

for all n. So the series $\sum b_n$ is convergent.

9.2 Q17

Let

$$a_n = \frac{(p+1)(p+2)...(p+n)}{(q+1)(q+2)...(q+n)}$$

Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{p+n+1}{q+n+1}$$

Then

$$\begin{aligned}\lim\left(n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)\right) &= \lim\frac{(q-p)n}{q+n+1}\\ &= q-p\end{aligned}$$

If $q > p+1$, then $q-p > 1$. Then $\sum a_n$ is convergent by the Raabe's test. Otherwise, $q-p \leq 1$. If $q-p < 1$, $\sum a_n$ is divergent by the Raabe's test. If $q-p = 1$, note that

$$a_n = \frac{p+1}{p+1+n}.$$

Then we can do limit comparison test with $b_n = \frac{1}{n}$:

$$\lim\left|\frac{a_n}{b_n}\right| = \lim\left|\frac{(p+1)n}{p+1+n}\right| = |p+1| = |q| > 0$$

Hence, we know that $\sum a_n$ is divergent by the limit comparison test.

9.3 Q1

(c) The n-th term $a_n = (-1)^{n+1}n/(n+2)$ of the series diverges because $\lim_n a_{2n} = -1$ while $\lim_n a_{2n+1} = 1$, and hence the series is divergent by the n-th term test.

(d) Let $f(x) = \ln x/x$. Since $f'(x) = (1-\ln x)/x^2 \leq 0$ for $x \geq e$, the sequence $(\ln n/n)$ is decreasing to zero from $n = 3$. So by the alternating series test, the series $\sum(-1)^{n+1}\ln n/n$ is convergent. However, it is not absolutely convergent, because $\ln n/n > 1/n$ for $n \geq 3$ and so the series $\sum \ln n/n$ diverges by a comparison test with the harmonic series $\sum 1/n$.

9.4 Q6

(a) Since $\limsup(|1/n^n|^{1/n}) = \lim 1/n = 0$, the radius of convergence is $R = +\infty$.

(c) Using Q5, since

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e^{-1},$$

the radius of convergence is $R = e^{-1}$.

9.4 Q11

By Taylor's theorem, for any $|x| < r$ and $n \in \mathbb{N}$, there exists $c \in [0, x]$ such that

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \right| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq B \cdot \frac{r^{n+1}}{(n+1)!},$$

using $|c| < r$. Since the sequence $a_n = r^n/n!$ converges to zero by ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{r}{n+1} = 0,$$

the right side of the above inequality converges to zero as n tends to ∞, which is to say that $\sum_{n=0}^{\infty} f^{(n)}(0)x^n/n!$ converges to $f(x)$ for $|x| < r$.