§6.4 Taylor's Theorem

<u>Recall</u>: If $f(x)$ has n-th derivative at a point x_0, then the polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the <u>n-th Taylor's Polynomial for f at x_0</u>.

Note: $P_n^{(k)}(x_0) = f^{(k)}(x_0) \quad \forall k = 0, 1, \cdots, n$.

<u>Thm 6.4.1</u> (<u>Taylor's Thm</u>)

Let
- $n \in \mathbb{N}$ (i.e. $n = 1, 2, \cdots$)
- $f : [a,b] \to \mathbb{R}$ such that $(a<b)$
- $f', \cdots, f^{(n)}$ are <u>continuous</u> on $[a,b]$ and
- $f^{(n+1)}$ exists on (a,b).

If $x_0 \in [a,b]$, then $\forall\, x \in [a,b]$, $\exists\, c$ <u>between x_0 and x</u> such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

where $P_n(x)$ is the n-th Taylor's Polynomial of f at x_0

Remark: $R_n(x) = f(x) - P_n(x)$ is referred as the remainder and

$$R_n(x) = \frac{f^{(n+1)}(c)}{n+1}(x-x_0)^{n+1}$$

is called the Lagrange form of the remainder,

or derivative form of the remainder.

Pf (of Thm 6.4.1)

Let $x_0, x \in [a,b]$ be given.

If $x_0 = x$, then the formula is clear.

If $x_0 \neq x$, we let

$J = [x_0, x]$ or $[x, x_0]$ depending on $x > x_0$ or $x_0 > x$.

Then J is a closed interval. $J \subset [a,b]$

Consider, for $t \in J$,

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2}f''(t) - \cdots - \frac{(x-t)^n}{n!}f^{(n)}(t)$$

Then, • $F(x) = 0$,

• $F(x_0) = f(x) - P_n(x) = R_n(x)$ is the remainder

And, by assumption, • $F(t)$ is continuous on J, and

• $F'(t)$ exists in the <u>interior</u> of J

with
$$
\begin{aligned}
F'(t) = &- f'(t) \\
&+ f'(t) - (x-t) f''(t) \\
&+ (x-t) f''(t) - \frac{(x-t)^2}{2} f^{(3)}(t) \\
&+ \cdots \\
&+ \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) - \frac{(x-t)^n}{n!} f^{(n+1)}(t) \\
= &- \frac{(x-t)^n}{n!} f^{(n+1)}(t) .
\end{aligned}
$$

Consider further the function

$$
G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0) \quad \text{for } t \in J
$$

Then G is continuous on J, differentiable in the <u>interior</u> of J,

and
$$
\begin{cases}
G(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = 0 \\
G(x) = F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = 0
\end{cases}
$$

By Rolle's Thm, $\exists$ $c \in$ interior of J (ie between x_0 & x)

s.t. $$0 = G'(c) = F'(c) + (n+1) \frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0)$$

$$\therefore \quad R_n(x) = F(x_0) = -\frac{1}{(n+1)} \cdot \frac{(x-x_0)^{n+1}}{(x-c)^n} F'(c)$$

$$= -\frac{1}{(n+1)} \cdot \frac{(x-x_0)^{n+1}}{(x-c)^n} \cdot \left(-\frac{(x-c)^n}{n!} f^{(n+1)}(c) \right)$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad \divideontimes$$

Applications of Taylor's Theorem

eg 6.4.2 (Approximation of values)

(a) Use Taylor's Thm with $n=2$ to approximate $\sqrt[3]{1+x}$, near $x=0$ $(x>-1)$.

Let $f(x) = (1+x)^{1/3}$, $x_0 = 0$.

For $n=2$, $P_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$

using $f(x) = (1+x)^{1/3}$, $f(0) = 1$;

$\Rightarrow \quad f'(x) = \frac{1}{3}(1+x)^{-2/3}$, $f'(0) = \frac{1}{3}$

$\Rightarrow \quad f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$, $f''(0) = -\frac{2}{9}$

$\therefore \quad P_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2$

And hence $\quad f(x) = P_2(x) + R_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_2(x)$

where $\quad R_2(x) = \frac{1}{3!} f'''(c)(x-x_0)^3 = \frac{1}{3!}\left(\frac{2\cdot 5}{9\cdot 3}\right)(1+c)^{-\frac{8}{3}} x^3$

$$= \frac{5}{81}(1+c)^{-\frac{8}{3}} x^3 \quad \text{for some } c \text{ between } 0 \,\&\, x.$$

Explicit eg: If $x = 0.3$.

Then $P_2(0.3) = 1 + \frac{1}{3}(0.3) - \frac{1}{9}(0.3)^2 = 1.09$

$$R_2(0.3) = \frac{5}{81} \cdot \frac{1}{(1+c)^{8/3}} (0.3)^3$$

$\Rightarrow$ $|R_2(0.3)| \leq \frac{5}{81}(0.3)^3$ since $c \in (0, 0.3) \Rightarrow c > 0$ $\Rightarrow 1+c > 1$

$$= \frac{1}{600} < 0.0017$$

$\therefore$ $|f(0.3) - P_2(0.3)| < 0.0017$

i.e $|\sqrt[3]{1.3} - 1.09| < 0.0017$

$\therefore$ $\sqrt[3]{1.3} \sim 1.09$ up to 2 decimal places.

(b) Use Taylor's Thm to approximate e with error $< 10^{-5}$ (5 decimal places)
(Assuming that we have defined e^x & proved $(e^x)' = e^x$, e^x increasing, and $e < 3$.)

Let $g(x) = e^x$, $x_0 = 0$.

Then by $(e^x)' = e^x$, we have $g^{(k)}(x) = e^x$, $\forall k = 1, 2, 3, \cdots$

Suppose that we need to use Taylor's Thm up to n.
Then the error is given by the remainder term

$$R_n(x) = \frac{1}{(n+1)!} e^c x^{n+1} \text{ for some } c \text{ between } 0 \ \& \ x.$$

Take $x=1$, we have

$$R_n(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$

Hence, to ensure error $< 10^{-5}$, we need

$$\frac{3}{(n+1)!} < 10^{-5}$$

i.e. $(n+1)! > 3\cdot 10^5 = 300000$ (Should use the smallest possible n to reduce calculation)

Try: $(8+1)! = 9! = 362880$ $\quad ((7+1)! = 8! = 40{,}320)$

$\therefore$ $n=8$ is the required value and hence

$$e = g(1) \approx P_8(1) = g(0) + g'(0)\cdot 1 + \frac{g''(0)}{2!}\cdot 1^2 + \cdots + \frac{g^{(8)}(0)}{8!}\cdot 1^8$$

$$= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!} \quad \text{with error} < 10^{-5}$$

$$= 2.718278\cdots \quad \text{(use calculator/computer)}$$

$\therefore$ $e = 2.71828$ up to 5 decimal places

eg 6.4.3 (Applications to inequalities)

(a) $1 - \frac{1}{2}x^2 \leq \cos x, \ \forall x \in \mathbb{R}$

Pf: Let $f(x) = \cos x$, $x_0 = 0$,

Then Taylor's Thm $\Rightarrow$

$$\cos x = 1 - \frac{1}{2}x^2 + R_2(x) \qquad \text{(check!)}$$

with $R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{\sin c}{6}x^3$ for some c between 0 & x.

If $0 \le x \le \pi$, then $0 \le c < \pi$ (the case $x=0$, we have $c=0$)

$\Rightarrow \sin c \ge 0$, $x^3 \ge 0$

Hence $R_2(x) \ge 0$.

$\therefore \quad 1 - \frac{1}{2}x^2 \le \cos x \quad \forall x \in [0, \pi]$.

If $x \in [-\pi, 0)$, then $y = -x \in (0, \pi]$

$\Rightarrow \quad 1 - \frac{1}{2}y^2 \le \cos y$

Using $\cos(-x) = \cos x$, we have $1 - \frac{1}{2}x^2 \le \cos x$. (check!)

Hence $1 - \frac{1}{2}x^2 \le \cos x$, $\forall x \in [-\pi, \pi]$.

If $|x| > \pi$, then $1 - \frac{1}{2}x^2 < 1 - \frac{1}{2}\pi^2 < -1 \le \cos x$

All together $1 - \frac{1}{2}x^2 \le \cos x \quad \forall x \in \mathbb{R}$ ⋕

(b) $\forall k = 1, 2, 3, \cdots$ & $\forall x > 0$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{1}{2k+1}x^{2k+1}$$

Pf: Let $f(x) = \ln(1+x)$ for $x > -1$ (mistake in textbook)

Then $f'(x) = \frac{1}{1+x}$, $f'' = \frac{-1}{(1+x)^2}$, $\cdots$ $f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$\Rightarrow$ nth Taylor's Poly of $\ln(1+x)$ at $x=0$ is

$$P_n(x) = 0 + 1\cdot x - \frac{1}{2!}\cdot x^2 + \frac{1}{3!}(2!)x^3 - \cdots + \frac{1}{n!}(-1)^{n-1}(n-1)!\,x^n$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n$$

and Remainder is (mistake in Textbook)

$$R_n(x) = \frac{(-1)^n\, n!}{(n+1)!}\,\frac{1}{(1+c)^{n+1}}\,x^{n+1}$$ for some c between 0 and x

If $x>0$, then $c>0$, and hence $1+c>1$

$\Rightarrow$

$$R_n(x) = \frac{(-1)^n}{(n+1)}\cdot\left(\frac{x}{1+c}\right)^{n+1} \begin{cases} >0 & \text{if } n \text{ even} \\ <0 & \text{if } n \text{ odd.} \end{cases}$$

$\therefore$ For $n=2k$,

$$\ln(1+x) = P_{2k}(x) + R_{2k}(x) > P_{2k}(x)$$

i.e. $\ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2k}}{2k}$ $\quad (\forall x>0)$

& For $n = 2k+1$

$$\ln(1+x) = P_{2k+1}(x) + R_{2k+1}(x) < P_{2k+1}(x)$$

ie $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1}$ $\quad (\forall x>0)$

$\divideontimes$

(c) $e^{\pi} > \pi^{e}$

Pf: Taylor's Thm

$\Rightarrow \quad e^{x} = 1 + x + R_1(x)$ (see eg 6.4.2)

with $R_1(x) = \dfrac{e^{c}}{2!} x^{2} > 0$ for some c between 0 & x.

(using the fact that $e^{c} > 0$, $\forall c \in \mathbb{R}$)

$\therefore \quad e^{x} > 1 + x, \quad \forall x \neq 0$

Put $x = \frac{\pi}{e} - 1 > 0$ (using known approx. values of π & e)

into it, we have

$$e^{\left(\frac{\pi}{e} - 1\right)} > 1 + \frac{\pi}{e} - 1 = \frac{\pi}{e}$$

$$\Rightarrow \quad e^{\frac{\pi}{e}} > \pi$$

$$\Rightarrow \quad e^{\pi} > \pi^{e} \quad \#$$

Application to Relative Extrema (Higher Derivative Test)

Thm 6.4.4 Let
- $f: I \to \mathbb{R}$, (I = interval)
- x_0 be an interior point of I
- $f', f'', \cdots, f^{(n)}$ exist and continuous in a nbd of x_0.
- $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$

Then

(i) n even & $f^{(n)}(x_0) > 0 \implies f$ has a relative minimum at x_0

(ii) n even & $f^{(n)}(x_0) < 0 \implies f$ has a relative maximum at x_0

(iii) n odd $\implies f$ has neither a relative minimum nor a relative maximum at x_0

Remark: If $n=2$, it is the 2nd Derivative Test.

Pf: If $f^{(n)}(x_0) \neq 0$ and $f^{(n)}$ continuous,

then $\exists$ nbd $U = (x_0 - \delta, x_0 + \delta) \subset I$ of x_0 such that

$$\operatorname{sgn}(f^{(n)}(x)) = \operatorname{sgn}(f^{(n)}(x_0)), \quad \forall x \in U. \quad \text{——} (*)$$

Now, using $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$,

the Taylor's Thm

$$\Rightarrow \quad f(x) = f(x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

$$= f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n, \quad \text{for some } c \text{ between } x_0 \text{ \& } x.$$

<u>Case (i)</u> n even, $f^{(n)}(x_0) > 0$.

By (*) & Taylor's, $\forall x \in U$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \geq 0$$

Since n even $\Rightarrow (x-x_0)^n \geq 0 \quad \forall x \in U$

$f^{(n)}(x_0) > 0 \Rightarrow f^n(c) > 0, \quad (x \in U \Rightarrow c \in U)$

$\therefore$ f has a relative minimum at x_0.

<u>Case (ii)</u> n even, $f^{(n)}(x_0) < 0$.

By (*) & Taylor's, $\forall x \in U$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \leq 0$$

Since n even $\Rightarrow (x-x_0)^n \geq 0 \quad \forall x \in U$

$f^{(n)}(x_0) < 0 \Rightarrow f^n(c) < 0, \quad (x \in U \Rightarrow c \in U)$

$\therefore$ f has a relative maximum at x_0.

Case (iii) n odd

Taylor's Thm $\Rightarrow$ $\forall x \in U$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \quad \text{changes sign}$$

Since n odd $\Rightarrow (x-x_0)^n$ change sign

$f^{(n)}(x_0) \neq 0 \Rightarrow f^n(c)$ has fixed sign $\quad (x \in U \Rightarrow c \in U)$

$\therefore$ Not maximum and also Not minimum.

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