§6.2 Cartan-Hadamard Thm

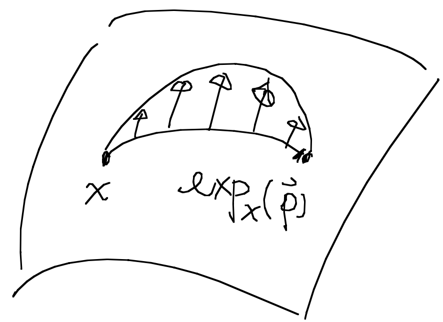
Lemma 6 $(d\exp_x)_{\vec{p}}$ is singular

$\Longleftrightarrow$ $\exists$ normal Jacobi field $\mathcal{V}(t)$ along

$\gamma(t) = \exp_x(t\vec{p})$

not identically zero s.t.

$\mathcal{V}(0) = \mathcal{V}(1) = 0$.



Pf: By the lemma right before the (original version) Gauss lemma in Ch 4,

$(d\exp_x)_{\vec{p}}$ is <u>non-degenerate</u> in the direction of $\vec{p}$.

Therefore, we only need to consider X s.t. $\langle X, \vec{p}\rangle = 0$.

Let $X \in T_x M \cong T_{\vec{p}}(T_x M)$ s.t. $\langle X, \vec{p}\rangle = 0$.

Then $\gamma_u(t) = \exp_x[t(\vec{p} + uX)]$

gives a <u>normal</u> Jacobi field (by (C))

with $U(0) = 0$, $U'(0) = X$.

Furthermore, $U(1) = (d\exp_x)_{\vec{p}}(X)$,

Therefore, if $X \in \ker((d\exp_x)_{\vec{p}})$ & $X \neq 0$, then $V(t)$ is the required normal Jacobi field.

This proves the direction "($\Rightarrow$)".

Conversely, any normal Jacobi field is the transversal vector field of a 1-para family of geodesics given by

$$\gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + u\,W(u)\right)\right]$$

with $\zeta(0) = \gamma(0)$, $\zeta'(0) = V(0)$

T, W = parallel vector fields along $\zeta(u)$.

Since $V(0) = 0$, we may take $\zeta(u) \equiv \gamma(0) = x$

$T = \vec{p}$ & $W = X = V'(0) \neq 0$, $\langle T, W\rangle = 0$.

Therefore,

$$0 = U(1) = (d\exp_x)_{\vec{p}}(X)$$

$$\Rightarrow 0 \neq X \in \ker((d\exp_x)_{\vec{p}})$$

$\therefore$ $(d\exp_x)_{\vec{p}}$ is singular $\divideontimes$

Def: If $(d\exp_x)_{\vec{p}}$ is singular, then $\vec{p}$ is called a conjugate point of the map $\exp_x$, and $\exp_x(\vec{p})$ is called a conjugate point of x along the geodesic $\gamma(t) = \exp_x(t\vec{p})$.

Thm 7 (Cartan–Hadamard)

(1) Let M be a complete Riemannian mfd. with nonpositive sectional curvature. Then $\forall x \in M$,

$\exp_x : T_xM \to M$ has no conjugate point;

(2) If M is a simply-connected complete Riem. mfd. s.t. for some $x \in M$, $\exp_x : T_xM \to M$ has no conjugate point, then $\exp_x : T_xM \to M$ is a diffeomorphism.

Pf of (1): Let U be a normal Jacobi field with $U(0)=0$ along a geodesic $\gamma:[0,\infty)\to M$.

Let $f(t)=\langle U(t),U(t)\rangle$ along γ, then

$$f' = 2\langle U',U\rangle$$

$$\Rightarrow\ f'' = 2\langle U',U'\rangle + 2\langle U'',U\rangle$$

$$= 2|U'|^2 - 2\langle R_{\gamma' U}\gamma', U\rangle$$

(Jacobi eqt.)

Since $\langle R_{\gamma' U}\gamma', U\rangle = K(\mathrm{span}(\gamma',U))\,|\gamma'\wedge U|^2$

≤ 0;

we have $f'' \geq 0$,

Now suppose $\gamma(t_0)$ is a conjugate point of x along some geodesic $\gamma : [0, \infty) \to M$. Then

Lemma 6 $\Rightarrow$ $\exists$ nontrivial normal Jacobi field $U(t)$ along γ s.t. $U(0) = U(t_0) = 0$ and $U(t) \not\equiv 0$ on $[0, t_0]$.

Applying the above, $|U(t)|^2$ is convex in t

$$\Rightarrow \quad 0 \leq |U(t)|^2 \leq \max\left(|U(0)|^2, |U(t_0)|^2\right) = 0$$
$$\forall t \in [0, t_0].$$

Contradiction! ※.

To prove (2), we need the following Lemmas:

Lemma 8: Let $\varphi: M \to N$ be a local isometry between (connected) Riemannian manifolds M & N. If M is complete, then N is complete and φ is a covering map.

Pf: Step 1: φ is surjective

- "φ = local isometry" $\Rightarrow$ $\varphi(M)$ open in N.
- Suppose $\gamma \subset N$ is a geodesic such that $\gamma \cap \varphi(M) \neq \emptyset$. Then $\exists$ $x \in M$ such that

$\varphi(x)$ is a point on γ. Since φ is a local isometry, then near the point x, $\varphi^{-1} \circ \gamma$ defines a geodesic segment in a nbd. of x in M (passing thro. the point x). Then the completeness of $M \Rightarrow \varphi^{-1} \circ \gamma$ extends to a $\tilde{\gamma} \subset M$ defined on $(-\infty, \infty)$. By assumption on φ, we have $\varphi \circ \tilde{\gamma} : (-\infty, \infty) \to \varphi(M) \subset N$ is a geodesic on N passing thro. $\varphi(x)$ and $\varphi \circ \tilde{\gamma} = \varphi \circ (\varphi^{-1} \circ \gamma) = \gamma$ in a nbd of $0 \in (-\infty, \infty)$
$(\gamma(0) = x)$

Therefore, uniqueness of geodesic $\Rightarrow$

$$\varphi \circ \tilde{\gamma} = \gamma$$

$$\Rightarrow \quad \gamma \subset \varphi(M).$$

So we have proved that for a geodesic segment γ in N s.t. $\gamma \cap \varphi(M) \neq \emptyset$, then $\gamma \subset \varphi(M)$. Now suppose y is a limiting point of $\varphi(M)$ in N, then $\exists\, x \in M$ s.t. $\exists$ a geodesic $\gamma(t)$, $t \in [0,1]$, in N such that

$$\gamma(0) = \varphi(x) \text{ and } \gamma(1) = y.$$

Therefore by the above argument, $y = \gamma(1) \subset \varphi(M)$.

$\therefore$ $\varphi(M)$ is closed in N.

Hence $\varphi(M)$ is both open & closed (non-empty) in a connected manifold N, we have

$$\varphi(M) = N.$$

$\Rightarrow$ φ is surjective.

<u>Note</u>: • In fact, we've proved the following commutative diagram:

$$\begin{tikzcd} T_x M \arrow[r, "d\varphi"] \arrow[d, "\exp_x^M"'] & T_{\varphi(x)} N \arrow[d, "\exp_{\varphi(x)}^N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

(local isom);

- and N is <u>complete</u>.

- Even more, (for $\delta > 0$ small s.t. exp is a diffeo when restricted to a ball of radius $< \delta$.)

we have

$$\begin{tikzcd}
B^M(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp^M_x"'] \arrow[dr, phantom, "\circlearrowleft"] & B^N(\delta) \arrow[d, "\exp^N_{\varphi(\cancel{y}x)}"] \\
B^M_\delta(x) \arrow[r, "\varphi"] & B^N_\delta(\varphi(\cancel{y}x))
\end{tikzcd}$$

<u>Step 2</u>: <u>φ is a covering map</u>.

i.e. We need to show that $\forall\, y \in N$, $\exists$ a nbd U of

y in N such that $\varphi^{-1}(U) = \bigcup_i W_i$ with

$$\begin{cases} \circ \; W_i \cap W_j = \emptyset \text{ for } i \neq j \\ \circ \; \varphi : W_i \to U \text{ is a diffeomaphism } \forall i. \end{cases}$$

<u>Pf of Step2</u>:

For any $y \in N$, $\exists \delta > 0$ such that

$$\exp_y^N : B^N(\delta) \longrightarrow B_\delta^N \quad \text{is a diffeomaphism}$$

where

$$B^N(\delta) = \{ v \in T_y N : |v|_N < \delta \}$$

$$B_\delta^N = \{ z \in N : d_N(z, y) < \delta \}.$$

Since φ is a local isom & hence a local diffeo,

$\varphi^{-1}(y)$ is a discrete set in M. Let

$$\varphi^{-1}(y) = \{x_i\}_{i\in\Lambda} \quad \text{for some index set } \Lambda.$$

and denote

$$B^i(\delta) = B^M(x_i, \delta) = \{v \in T_{x_i}M : |v|_M < \delta\}$$

$$B^i_\delta = B^M_\delta(x_i) = \{\xi \in M : d_M(\xi, x_i) < \delta\}.$$

<u>Claim</u>: (i) $\varphi^{-1}(B^N_\delta) = \bigcup_i B^i_\delta$

(ii) $\forall i$: $\varphi : B^i_\delta \longrightarrow B^N_\delta$ is a diffeo.

(iii) $\forall i \neq j$, $B^i_\delta \cap B^j_\delta = \emptyset$.

Pf of (i) : It is clear that $\bigcup_i B_\delta^i \subset \varphi^{-1}(B_\delta^N)$

since φ is a local isometry.

Conversely, for $\varphi^{-1}(B_\delta^N) \subset \bigcup_i B_\delta^i$, we take a

$z \in \varphi^{-1}(B_\delta^N)$. Then $\varphi(z) \in B_\delta^N$. By construction

of B_δ^N, $\exists$ unique geodesic $\gamma : [0,1] \to B_\delta^N$

s.t. $\gamma(0) = \varphi(z)$ & $\gamma(1) = y$.

Then by the above argument (in proof of step 1),

$\exists$ a geodesic $\tilde{\gamma} : [0,1] \to M$ s.t.

$\tilde{\gamma}(0) = z$ & $\varphi(\tilde{\gamma}(t)) = \gamma(t)$, $\forall t$

$\Rightarrow \quad \varphi(\tilde{\gamma}(1)) = \gamma(1) = y$

$\Rightarrow \quad \tilde{\gamma}(1) \in \varphi^{-1}(y) = \{x_i\}_{i \in \Lambda}$.

$\Rightarrow \quad \tilde{\gamma}(1) = x_i$ for some $i \in \Lambda$.

Again, φ = local isom $\Rightarrow$

$$\text{Length}_M(\tilde{\gamma}) = \text{Length}_N(\gamma) < \delta$$

$\Rightarrow \quad \tilde{\gamma}(0) = z$ has a distance $< \delta$ to x_i

i.e. $\quad z \in B^i_\delta \subset \bigcup_i B^i_\delta$.

This completes the proof of (i).

Pf of (ii) : By the note in Step 1, & φ is a local isom., we have

$$\begin{tikzcd} B^i(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp^M_{x_i}"'] \arrow[dr, phantom, "\circlearrowright"] & B^N(\delta) \arrow[d, "\exp^N_y"] \\ B^i_\delta \arrow[r, "\varphi"] & B^N_\delta \end{tikzcd}$$

i.e.

$$\varphi \circ \exp^M_{x_i} = \exp^N_y \circ d\varphi$$

By the choice of $\delta > 0$, $\exp^N_y$ and $d\varphi$ are diffeomorphisms. Hence $\exp^M_{x_i}$ has to be an immersion. On the other hand $\exp^M_{x_i} : B^i(\delta) \to B^i_\delta$

is surjective (by completeness of M), therefore we have

$$\varphi = \exp^N_y \circ d\varphi \circ (\exp^M_{x_i})^{-1}$$

which is a diffeomorphism. This proves (ii).

<u>Pf of (iii)</u>: Let $i \neq j \in \Lambda$. Suppose that $\tilde{B}^i_\delta \cap \tilde{B}^j_\delta \neq \phi$.

Then $\exists\ \xi \in \tilde{B}^i_\delta \cap \tilde{B}^j_\delta$. Using (ii), $\exists$ geodesics

$\tilde{\gamma}_i \in \tilde{B}^i_\delta$ & $\tilde{\gamma}_j \subset \tilde{B}^j_\delta$ joining ξ to x_i & x_j respectively.

Then $\varphi(\tilde{\gamma}_i)$ & $\varphi(\tilde{\gamma}_j)$ are geodesics in B^N_δ joining $\varphi(\xi)$ and $\varphi(x_i) = y = \varphi(x_j)$.

$\Rightarrow \quad \varphi(\tilde{\gamma}_i) = \varphi(\tilde{\gamma}_j) = \gamma$ the unique geodesic in B_δ^N joining $\varphi(\zeta)$ to y.

Therefore $\tilde{\gamma}_i, \tilde{\gamma}_j$ are both liftings of γ passing thro. a common point $\cancel{z}$ ζ, we have $\tilde{\gamma}_i = \tilde{\gamma}_j$.

$\Rightarrow \quad x_i = \tilde{\gamma}_i(1) = \tilde{\gamma}_j(1) = x_j$ Contradiction!

This completes the proof of (iii) $\#$

By the claim, B_δ^N is the required unifam nbd of y.

$\therefore \varphi$ is covering. $\#$

Lemma 9: Let
- M = complete Riemannian mfd.
- $x \in M$ s.t.,
- $\exp_x : T_xM \to M$ has no conjugate point.

Then $\exp_x$ is a covering map.

Pf: Let g = Riem. metric of M.

Denote $\tilde{g} = (\exp_x)^* g$ be the pull-back metric of g by $\exp_x$ on T_xM. ($\tilde{g}$ is a metric since $\exp_x$ has no conjugate point.)

Claim: $\tilde{g}$ is a complete metric on T_xM.

Pf of Claim: Note that Euclidean rays (from 0) in T_xM can be parametrized by

$$\tilde{\gamma} : [0, \infty) \longrightarrow T_xM \qquad \text{for some } v \in T_xM$$
$$t \longmapsto tv$$

By definition of $\exp_x$, $\exp_x(\tilde{\gamma}(t))$ is a geodesic in M starting at x. Therefore, by definition of $\tilde{g} = (\exp_x)^* g$, $\tilde{\gamma}(t)$ is a geodesic of $\tilde{g}$ starting from 0. This implies geodesics from $0 \in T_xM$ is defined $\forall\, t \in [0, \infty)$. Hence $\exp_0 : T_0(T_xM) \to (T_xM, \tilde{g})$ is defined on the whole $T_0(T_xM)$. Therefore, Hopf-Rinow Thm $\Rightarrow$ $(T_xM, \tilde{g})$ is complete. This completes the proof of the claim. $\square$

Now by the claim and the assumption that $\exp_x$ has no conjugate point,

$$\exp_x : (T_xM, \tilde{g}) \to (M, g)$$

is a local isometry from a complete Riem. mfd.

$\therefore$ Lemma 8 $\Rightarrow$ $\exp_x : T_xM \to M$ is a covering

$\#$

Pf of (2) of Cartan-Hadamard:

By Lemma 9, $\exp_x : T_xM \to M$ is a covering.

Together with the assumption that M is simply-connected, we have proved that $\exp_x$ is a diffeomorphism. #

Thm 10: Let M, N = Simply-connected n-dim'l space forms with constant sectional curvature K.

Let $x \in M$ & $y \in N$ and

$\{e_1, \cdots, e_n\} \subset T_x M$ & $\{\varepsilon_1, \cdots, \varepsilon_n\} \subset T_y N$

are orthonormal basis respectively.

Then $\exists$ unique isometry $\varphi : M \to N$ such that

$$\begin{cases} \varphi(x) = y \quad \text{and} \\ d\varphi(e_i) = \varepsilon_i \quad \forall i \end{cases}.$$

Note: Thm 10 $\Rightarrow$ uniqueness of the Thm 1 of Ch5.

To prove this Thm, we need the following Lemmas:

Lemma 11: Let
- M = n-dim'l space form
- constant sectional curvature K

• $x \in M$, $\{e_1, \cdots, e_n\} \subset T_xM$ orthn. basis.

Then the curvature tensor satisfies

$$R_{e_i e_j} e_k = K(\delta_{ik} e_j - \delta_{jk} e_i), \quad \forall i, j, k = 1, \cdots, n.$$

Pf: Define $\tilde{R}$ by the ~~L~~R.H.S. ie

$$\tilde{R}_{e_i e_j} e_k \overset{\text{def}}{=} K(\delta_{ik} e_j - \delta_{jk} e_i)$$

Then $\tilde{R}$ can be extended to a tensor (Ex) satisfying all the symmetric properties of the curvature tensor (i.e. (1)–(4) in Lemma 1 of §3.3)

(Ex). Furthermore, for tangent vectors v & w with $|v| = |w| = 1$ & $\langle v, w \rangle = 0$, one has $\langle \tilde{R}_{vw} v, w \rangle = K$. (Ex)

Therefore (Lemma 2 of §3.3), we have

$$\tilde{R} \equiv R. \quad ※$$

Lemma 12: Same assumption as in Lemma 11.

Let
- $v \in T_x M$ with $|v| = 1$
- $v^{\perp}$ = orthogonal complement of v.

Then $$R_{vw}v = \begin{cases} kw, & \text{if } w \in v^{\perp} \\ 0, & \text{if } w = cv \text{ for some } c \in \mathbb{R} \end{cases}$$

(Pf: Straight forward from Lemma 11)