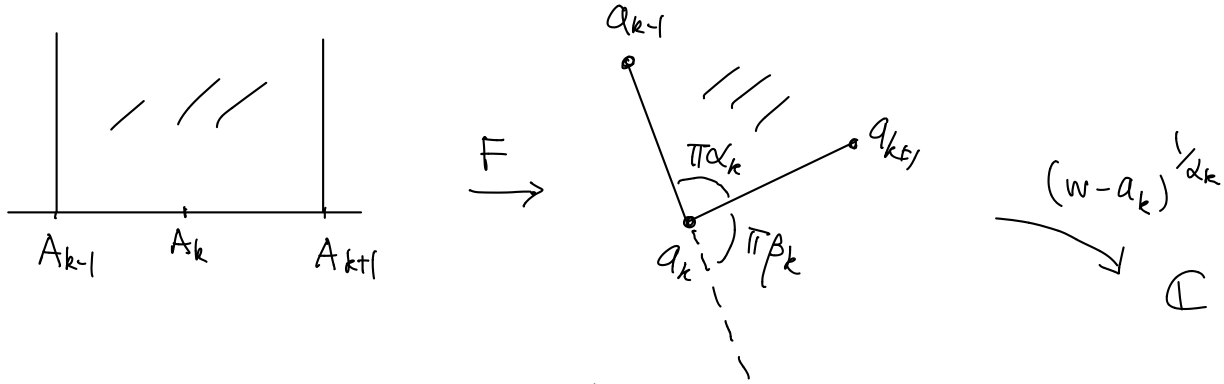
Pf. Step 1 <u>Local behavior of F''/F' at $z = A_k$.</u>



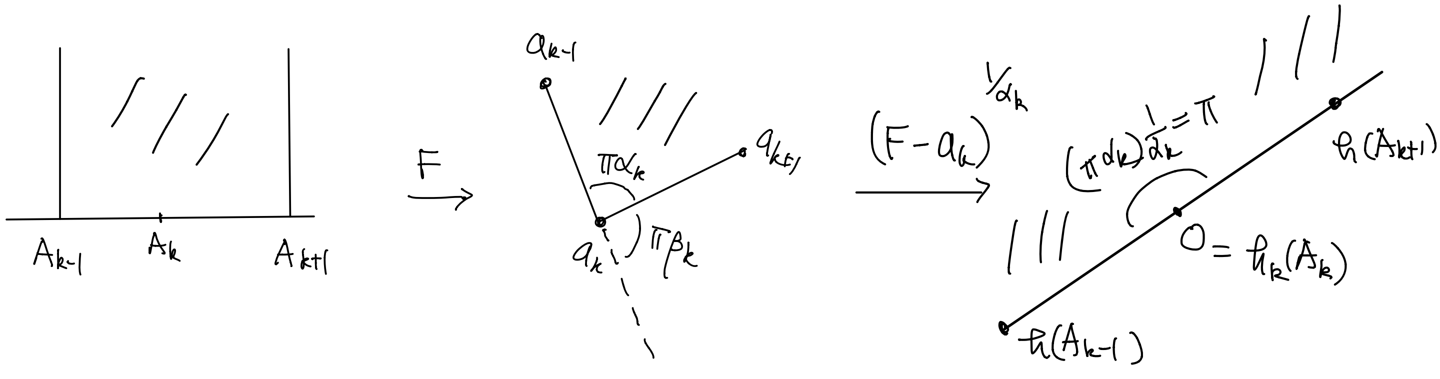
Consider $h_k(z) = [F(z) - a_k]^{1/\alpha_k}$ for $k = 2, \cdots, n-1$

Since $\{ A_{k-1} < x < A_{k+1},\ y > 0 \}$ is simply-connected
and $F(z) \neq a_k$ ($\forall z$ inside),
$h_k(z)$ is well-defined by choosing a branch of log.

$1/\alpha_k > 0$ and $F(z)$ continuous up to boundary including $z = A_k$,
we see that $h_k(z)$ extends to the horizontal line segment (A_{k-1}, A_k).
Note that the value of the extended $h_k(z)$ at $z = A_k$ is $h_k(A_k) = 0$ as

$$|h_k(z)| = |F(z) - a_k|^{1/\alpha_k} \to 0 \text{ as } z \to A_k$$

(which shows the continuity at $z = A_k$ too.)

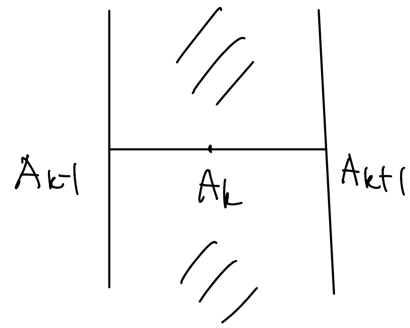


And the image of the horizontal arc (A_{k-1}, A_k) is a straight line segment by the property of the map $z \mapsto z^{1/\alpha_k}$.

Hence composing with a rotation of a suitable angle θ_k, $e^{i\theta} h_k$ maps (A_{k-1}, A_k) into $\mathbb{R}$.

Schwarz reflection principle $\Rightarrow$ $e^{i\theta} h_k$ and hence h_k can be analytically continuated to the infinite strip

$\{z = x+iy : A_{k-1} < x < A_{k+1}\}$.



__Claim__: $h_k'(z) \neq 0$, $\forall z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\}$.

Note that for $z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\} \cap \mathbb{H}$

$$\frac{h_k'(z)}{h_k(z)} = \frac{1}{\alpha_k} \frac{F'(z)}{F(z) - a_k}$$

$F : \mathbb{H} \to P$ is conformal $\Rightarrow$ $F'(z) \neq 0$, $\forall z \in \mathbb{H}$.

Hence $h_k'(z) \neq 0$, $\forall z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\} \cap \mathbb{H}$.

Recall that, the Schwarz reflection principle is constructed using $\overline{e^{i\theta} h(\bar{z})}$. So up to a multiple of non zero constant, $h'(z)$ for $z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}, \ \mathrm{Im}\, z < 0\}$ is $\overline{h'(\bar{z})}$ and hence $h'(z) \neq 0$.

It remains to consider $z = x \in (A_{k-1}, A_k)$.

Note that $F\big|_{(A_{k-1}, A_k)}$ is injective, $h_k\big|_{(A_{k-1}, A_k)}$ is also injective. Hence $\frac{\partial}{\partial x} h_k(x) \neq 0 \quad \forall x \in (A_{k-1}, A_k)$. (cpx derivative)

Since h_k is holo in a nbd of $z = x$, $h_k'(x) \neq 0$.

We've shown that $h_k'(z) \neq 0, \ \forall z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\}$.

For $z \in \{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\} \cap \mathbb{H}$,

$$F'(z) = \alpha_k \cdot (F(z) - a_k) \cdot \frac{h_k'(z)}{h_k(z)}$$

$$= \alpha_k \, h_k(z)^{\alpha_k - 1} h_k'(z) = \alpha_k \, h_k(z)^{-\beta_k} h_k'(z)$$

$\Rightarrow$

$$\frac{F''(z)}{F'(z)} = -\beta_k \frac{h_k'(z)}{h_k(z)} + \frac{h_k''(z)}{h_k'(z)}$$

Since $h_k'(z) \neq 0$, $\frac{F''(z)}{F'(z)}$ extended to a <u>meromorphic</u> function in $\{A_{k-1} < \mathrm{Re}(z) < A_{k+1}\}$ with a <u>simple pole</u> at the only zero $z = A_k$ of $h_k(z)$.

(clear from the definition of h_k on the upper half strip and the reflection principle)

By Taylor's expansion of $h_k(z)$ near $z = A_k$, the

$$\mathrm{Res}_{z=A_k} \frac{F''(z)}{F'(z)} = -\beta_k .$$

$$\therefore \quad \frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z-A_l} = \left[\beta_k \cdot \left(\frac{1}{z-A_k} - \frac{h_k'(z)}{h_k(z)} \right) + \frac{h''(z)}{h'(z)} \right] + \sum_{l \neq k} \frac{\beta_l}{z-A_l}$$

$$= E_k(z)$$

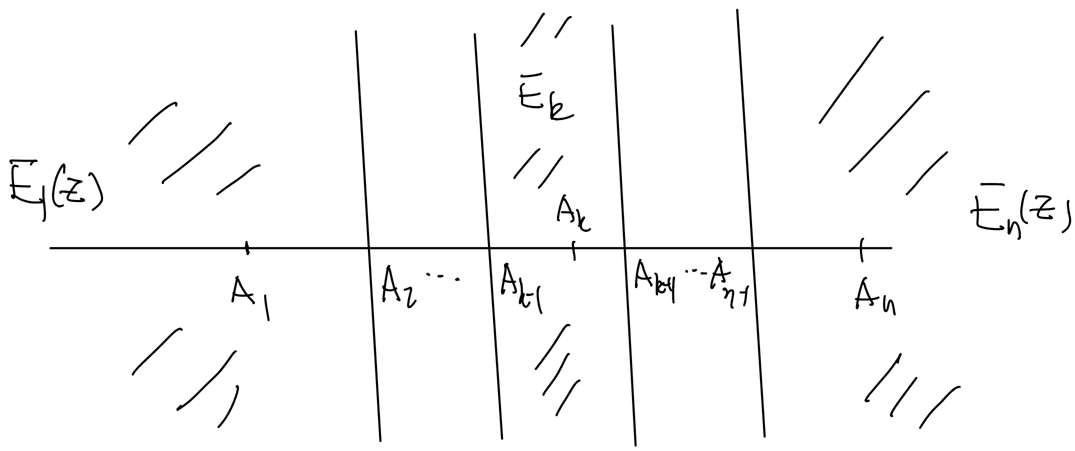
with $E_k(z)$ is holo in $\{A_{k-1} < Re(z) < A_k\}$ (as $\frac{1}{z-A_l}$, $l \neq k$, are also holo in the domain)

Similarly, there exists $E_1(z)$ holo in $\{-\infty < Re(z) < A_2\}$ such that

$$\frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z-A_l} = E_1(z), \quad \forall z \in \{-\infty < Re(z) < A_2\},$$

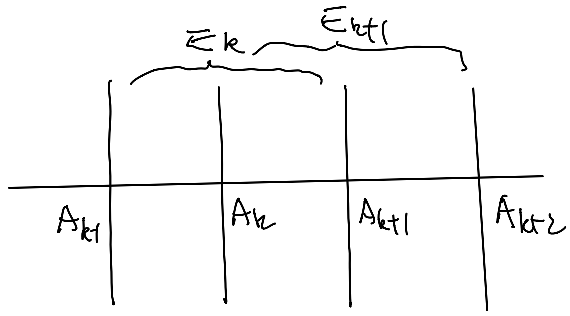
and $E_n(z)$ holo in $\{A_{n-1} < Re(z) < \infty\}$ such that

$$\frac{F''(z)}{F'(z)} + \sum_{l=1}^{n} \frac{\beta_l}{z-A_l} = E_n(z), \quad \forall z \in \{A_{n-1} < Re(z) < \infty\},$$



Step 2 Global behavior of $F''(z)/F'(z)$

Note that the domains of E_k & E_{k+1} overlaps on $\{A_k < \mathrm{Re}(z) < A_{k+1}\}$, E_k & E_{k+1} together define an analytic function on $\{A_{k-1} < \mathrm{Re}(z) < A_{k+2}\}$. (and agree on $\mathbb{H}$)



And so on, $E_1, \cdots, E_n$ all together defines an entire function $E(z)$ (on $\{-\infty < \mathrm{Re}(z) < +\infty\}$).

This implies,

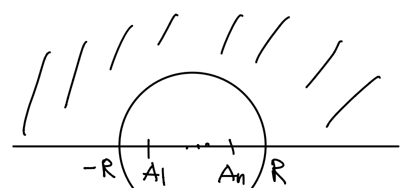
$$\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \quad \text{is entire}$$

(More precisely, extends to an entire function)

Step 3: Estimate of $F''(z)/F'(z)$ at $|z| \to \infty$.

Note that by uniqueness of analytic continuation, the extension of $\dfrac{F''(z)}{F'(z)}$ given above is equal to the extension given below:

Consider $R > \max\{|A_1|, |A_n|\}$, then $F(z)$ is holomorphic on $\mathbb{H} \cap \{|z| > R\}$.



Then Thm 4.2 (and similar argument as in Prop 4.1 (i))

$\Rightarrow$ $F(z)$ maps $(-\infty,-R)\cup(R,+\infty)$

into the straight line segment (a_n, a_1) (an edge of P).
Hence one can apply Schwarz reflect principle as before to extend $F(z)$ analytically to

$$\{ z : |z| > R \} .$$

Moreover, similar argument as in the proof of $\tilde{h}_k'(z) \neq 0$ before, we have F conformal $\Rightarrow$ $F'(z) \neq 0$ $\forall z \in \{ |z| > R \}$.

And hence $F''(z)/F'(z)$ is well-defined on $\{|z|>R\}$

Since this extension coincides with the previous one on $\{|z|>R\} \cap \mathbb{H}$, they are identical.

Now, by Laurent expansion on $\{|z|>R\}$

$$F(z) = C_0 + \frac{C_k}{z^k} + \frac{C_{k+1}}{z^{k+1}} + \cdots \quad \text{with } C_k \neq 0, \; k \geq 1$$

and $C_0 = F(\infty) \in P$.

$$\Rightarrow \begin{cases} F'(z) = -\dfrac{kC_k}{z^{k+1}} - \dfrac{(k+1)C_{k+1}}{z^{k+2}} + \cdots \\[2ex] F''(z) = \dfrac{k(k+1)C_k}{z^{k+2}} + \dfrac{(k+1)(k+2)C_{k+1}}{z^{k+3}} + \cdots \end{cases}$$

$$\Rightarrow \frac{F''(z)}{F'(z)} = \frac{-(k+1)}{z} \cdot \frac{\left(1 + \frac{k+2}{k} \cdot \frac{C_{k+1}}{C_k} \cdot \frac{1}{z} + \cdots\right)}{\left(1 + \frac{k+1}{k} \cdot \frac{C_{k+1}}{C_k} \cdot \frac{1}{z} + \cdots\right)} \quad \text{for } \{|z|>R\}$$

$$\Rightarrow \frac{|F''(z)|}{|F'(z)|} \to 0 \quad \text{as } |z| \to \infty .$$

Final Step :

By Steps 2 & 3 , the entire function

$$\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k} \to 0 \quad \text{as } |z| \to \infty .$$

Liouville's Thm $\Rightarrow$ $\displaystyle \frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k} = 0 .$

Let $\displaystyle Q(z) = \frac{1}{(z-A_1)^{\beta_1} \cdots (z-A_n)^{\beta_n}} = S'(z)$

Then $\forall z \in \mathbb{H}$,

$$\frac{d}{dz}\left(\frac{F'(z)}{Q(z)}\right) = \frac{F'(z)}{Q(z)}\left[\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k}\right] = 0$$

$\Rightarrow$ $F'(z) = C_1 Q(z)$ $\quad \forall z \in \mathbb{H}$, for some constant C_1 $(\neq 0)$

Hence

$F(z) = C_1 S(z) + C_2$, $\forall z \in \mathbb{H}$, for some const. C_2 .

$\divideontimes$

Note that in Thm 4.6, $F(\infty)$ <u>can't</u> be a vertex of $\mathcal{P}$.

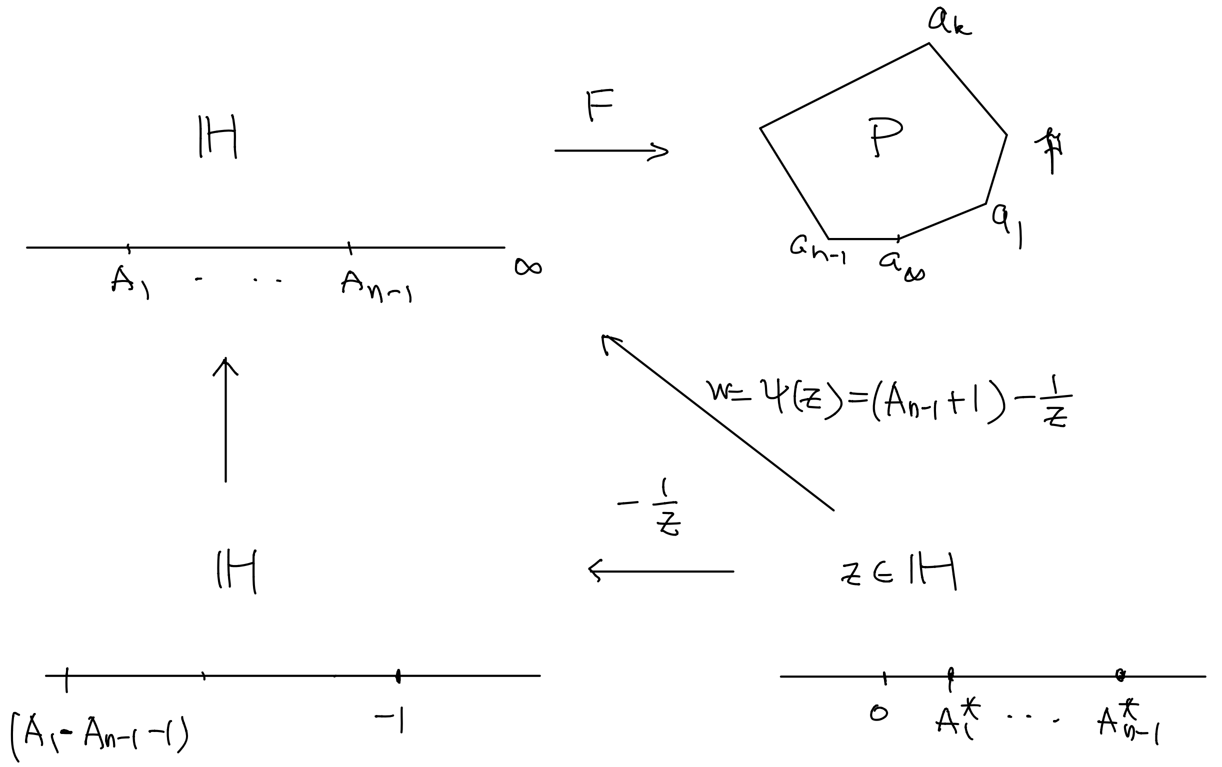
If $F(\infty)$ is a vertex of $\mathcal{P}$, then the formula actually simpler with only $(n-1)$-degree in the denominator.

With the same notations as before, we have

Thm 4.7 If $F: \mathbb{H} \to P$ conformal and maps $A_1 < \cdots < A_{n-1}, \infty$ to the vertices of $\mathcal{P}$, then $\exists$ (cpx) constants C_1 and C_2 such that

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2$$

Pf:



Define $\psi(z) = (A_{n-1}+1) - \frac{1}{z} = \dfrac{(A_{n-1}+1)z + (-1)}{1 \cdot z + 0}$

with $A_{n-1}+1, -1, 1, 0 \in \mathbb{R}$ and $(A_{n-1}+1)\cdot 0 - (-1)\cdot 1 = 1$

$\therefore \psi \in \mathrm{Aut}(\mathbb{H})$.

Moreover $z = \psi^{-1}(w) = \dfrac{1}{(A_{n-1}+1) - w}$

Let $A_k^* = \psi^{-1}(A_k) = \dfrac{1}{(A_{n-1}+1) - A_k} = \dfrac{1}{(A_{n-1} - A_k) + 1} > 0$ and $\neq \infty$.

since $(A_{n-1} - A_k) + 1 \geq 1$, $\forall k = 1, \cdots, n-1$.

Also $$A_k^* - A_{k-1}^* = \frac{1}{(A_{n-1}+1) - A_k} - \frac{1}{(A_{n-1}+1) - A_{k-1}}$$

$$= \frac{A_k - A_{k-1}}{(A_{n-1} - A_k + 1)(A_{n-1} - A_{k-1} + 1)} > 0$$

For $w = \infty$, we have
$$0 = \Psi^{-1}(\infty)$$

Hence $F \circ \Psi : \mathbb{H} \to P$ conformal and maps $0 < A_1^* < \cdots < A_{n-1}^*$ to $a_\infty, a_1, \cdots, a_{n-1}$ of P (still in order)

with exterior angles $\beta_\infty, \beta_1, \cdots, \beta_{n-1}$ satisfying
$$\beta_\infty + \sum_{k=1}^{n-1} \beta_k = 2 .$$

Applying Thm 4.6, $\exists$ constants c_1' & c_2' such that

$$F \circ \Psi(z) = c_1' \int_0^z \frac{d\zeta}{\zeta^{\beta_\infty} (\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + c_2'$$

$$\Rightarrow F(w) = c_1' \int_0^{\Psi^{-1}(w)} \frac{d\zeta}{\zeta^{\beta_\infty} (\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + c_2'$$

$$= c_1' \left(\int_{\Psi^{-1}(0)}^{\Psi^{-1}(w)} + \int_0^{\Psi^{-1}(0)} \right) \frac{d\zeta}{\zeta^{\beta_\infty} (\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + c_2'$$

Note that $\Psi^{-1}(0) = \infty$. Since the integral converges, the formula is valid (in fact, $C_1' \int_0^{\psi^{-1}(0)} \frac{d\zeta}{\zeta^{\beta_\infty} \prod_{k=1}^{n-1} (\zeta - A_k^*)^{\beta_k}} + C_2'$ converges

to a point on $\mathbb{P}$).

$$\therefore \quad F(w) = C_1' \int_\infty^{\psi^{-1}(w)} \frac{d\zeta}{\zeta^{\beta_\infty} (\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_{n-1}^*)^{\beta_{n-1}}} + C_2$$

where $C_2 = C_1' \int_0^{\psi^{-1}(0)} \frac{d\zeta}{\zeta^{\beta_\infty} \prod_{k=1}^{n-1} (\zeta - A_k^*)^{\beta_k}} + C_2'$

Now substitute $\quad \varphi = \psi(\zeta) = (A_{n-1}+1) - \frac{1}{\zeta}$

Then $\quad \zeta = \frac{1}{A_{n-1}+1-\varphi} \quad$ and $\quad d\zeta = \frac{d\varphi}{(A_{n-1}+1-\varphi)^2}$.

$$\Rightarrow F(w) = C_2 + C_1' \int_0^w \frac{1}{\psi^{-1}(\varphi)^{\beta_\infty} \prod_{k=1}^{n-1} (\psi^{-1}(\varphi) - A_k^*)^{\beta_k}} \cdot \frac{d\varphi}{(A_{n-1}+1-\varphi)^2}$$

Note that $\psi^{-1}(\varphi) - A_k^* = \psi^{-1}(\varphi) - \psi^{-1}(A_k)$

$$= \frac{1}{A_{n-1}+1-\varphi} - \frac{1}{A_{n-1}+1-A_k}$$

$$= \frac{\varphi - A_k}{(A_{n-1}+1-\varphi)(A_{n-1}+1-A_k)}$$

$$\therefore F(w) = C_2 + C_1' \int_0^w \frac{(A_{n-1}+1-\varphi)^{\beta_\infty} \prod_{k=1}^{n-1} [(A_{n-1}+1-\varphi)(A_{n-1}+1-A_k)]^{\beta_k}}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k} \cdot (A_{n-1}+1-\varphi)^2} d\varphi$$

Using $2 = \beta_\infty + \sum_{k=1}^{n-1} \beta_k$,

$$F(W) = C_2 + C_1' \cdot \prod_{k=1}^{n-1} (A_{n-1} - A_k + 1)^{\beta_k} \cdot \int_0^W \frac{d\varphi}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k}}$$

$$= C_1 \int_0^W \frac{d\varphi}{\prod_{k=1}^{n-1} (\varphi - A_k)^{\beta_k}} + C_2 ,$$

where $\quad C_1 = C_1' \cdot \prod_{k=1}^{n-1} (A_{n-1} - A_k + 1)^{\beta_k}$,

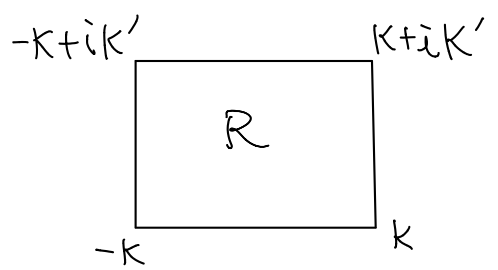
which is the desired formula.

4.5 Return to Elliptic Integrals

In Eg 3 of section 4.1, it was shown that the elliptic integral

$$I(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \qquad (0 < k < 1)$$

maps $\mathbb{R}$-axis to the boundary of the rectangle:



where $\quad K = K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

$$K' = K'(k) = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}} .$$

However, we mentioned that we haven't proved that $I(z)$ maps $\mathbb{H}$ conformally onto R. Now, we can show it by Thm 4.6 as follows. For this purpose, we need a lemma

Lemma (Ex 15 of the Textbook on pg 251)

(a) If $\Phi \in Aut(\mathbb{H})$ and $\exists$ distinct points A_1, A_2, A_3 on $\mathbb{R}$-axis such that $\Phi(A_i) = A_i$, for $i=1,2,3$. Then $\Phi = Id_{\mathbb{H}}$

(b) Let $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$ ($x_i, y_i \in \mathbb{R}$). Then $\exists\ \Phi \in Aut(\mathbb{H})$ such that

$$\Phi(x_i) = y_i, \quad i = 1, 2, 3.$$

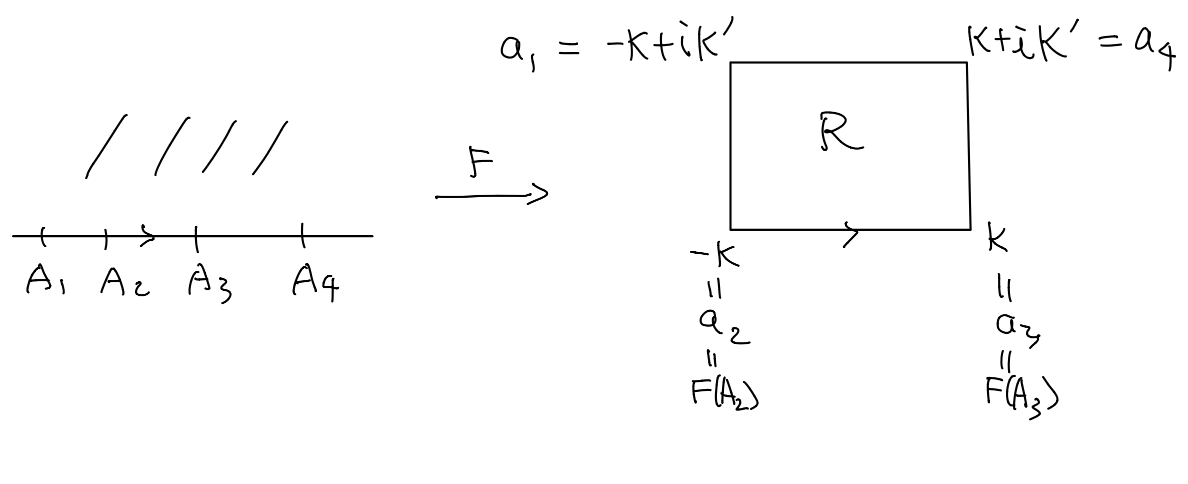
Same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$

(Pf: Ex!)

Pf of $I(z): \mathbb{H} \to R$ conformal

Let $F: \mathbb{H} \to R$ be a conformal map (existence by Riemann Mapping)

Let $A_1 < A_2 < A_3 < A_4$ (A_4 may $= \infty$) be points that maps to the vertices $-K, K, K+iK', -K+iK'$



By Thm 4.6, F maps $[A_2, A_3]$ to $[-K, K]$.

Hence $A_2 < F^{-1}(0) < A_3$

By lemma above, $\exists\ \Phi \in \mathrm{Aut}(\mathbb{H})$ such that

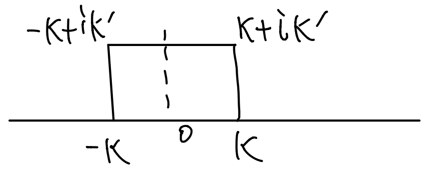
$$\Phi(-1) = A_2,\quad \Phi(0) = F^{-1}(0),\quad \Phi(1) = A_3.$$

$\Rightarrow G := F \circ \Phi : \mathbb{H} \to R$ conformal and satisfies

$$\begin{cases} G(-1) = -K \\ G(0) = 0 \\ G(1) = K \end{cases}.$$

Then note that the upper-half plane $\mathbb{H}$ and the rectangle R are symmetric wrt

$$x+iy = z \mapsto -\bar{z} = -x+iy$$



$$G^*(z) = -\overline{G(-\bar{z})} : \mathbb{H} \xrightarrow{z \mapsto -\bar{z}} \mathbb{H} \xrightarrow{G} R \xrightarrow{w \mapsto -\bar{w}} R$$

Cauchy-Riemann equation (& Chain rule)

$\Rightarrow G^* : \mathbb{H} \to R$ is also conformal.

(Typo in the Textbook.)

Hence $G^{-1} \circ G^* : \mathbb{H} \to \mathbb{H} \in \mathrm{Aut}(\mathbb{H})$.

Observe
$$G^*(1) = -\overline{G(-1)} = K = G(1)$$
$$G^*(-1) = -\overline{G(1)} = -K = G(-1)$$
$$G^*(0) = -\overline{G(0)} = 0 = G(0).$$

The Lemma $\Rightarrow G^{-1} \circ G^* = \mathrm{Id}_{\mathbb{H}}$, i.e. $G = G^*$.

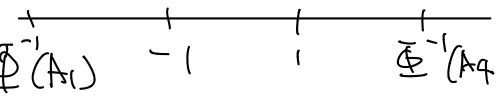
$$K+iK' = F(A_4) = G(\Phi^{-1}(A_4)) = G^*(\Phi^{-1}(A_4)) = -\overline{G(-\overline{\Phi^{-1}(A_4)})}$$

$$\Rightarrow \quad G(-\Phi^{-1}(A_4)) = \overline{(-K-iK')} = -K+iK' = F(A_1) = G(\Phi^{-1}(A_1))$$

By injectivity, $\Phi^{-1}(A_1) = -\Phi^{-1}(A_4)$.

Together with the orientation, we must have

$$\Phi^{-1}(A_4) > 1$$



And hence $\exists\ \ell \in (0,1)$ s.t. $\Phi^{-1}(A_4) = \frac{1}{\ell}$.

Altogether, we may assume the map $F : \mathbb{H} \to P$ and points A_1, A_2, A_3, A_4 at the beginning of the proof satisfies

$$F(0) = 0 \quad \text{and}$$

$$A_1 = -\frac{1}{\ell}, \quad A_2 = -1, \quad A_3 = 1, \quad A_4 = \frac{1}{\ell} \quad (0 < \ell < 1)$$

By Thm 4.6, $\exists\ C_1, C_2$ such that

$$F(z) = C_1 \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-\ell^2\zeta^2)}} + C_2$$

(Note that more precisely, $c_1' \int_0^z \frac{d\zeta}{\sqrt{(\zeta+\frac{1}{\ell})(\zeta+1)(\zeta-1)(\zeta-\frac{1}{\ell})}} + c_2$ $\therefore c_1 = \ell c_1'$)

Using $F(0) = 0$, we have $C_2 = 0$

Putting $z = 1, \frac{1}{\ell}$ in the formula, we have

$$K(k) = K = C_1 \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-\ell^2\zeta^2)}} = C_1 K(\ell)$$

and

$$K+iK' = F\left(\frac{1}{\ell}\right) = C_1\left[K(\ell) + \int_1^{\frac{1}{\ell}} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-\ell^2\zeta^2)}}\right]$$

$$= C_1 K(l) + C_1 i \int_1^{\frac{1}{l}} \frac{dx}{\sqrt{(x^2-1)(1-l^2x^2)}}$$

$$\Rightarrow \quad K'(k) = C_1 K'(l)$$

By Ex. 24 of the Textbook,

$$K'(k) = K(\sqrt{1-k^2})$$

$\therefore$ we have $\begin{cases} K(k) = C_1 K(l) \\ K(\sqrt{1-k^2}) = C_1 K(\sqrt{1-l^2}) \end{cases}$

$$\Rightarrow \quad \frac{K(k)}{K(\sqrt{1-k^2})} = \frac{K(l)}{K(\sqrt{1-l^2})}$$

Clearly $K(k)$ is strictly increasing in k, $(0<k<1)$ (Ex!)

$\Rightarrow$ $K(\sqrt{1-k^2})$ is strictly decreasing in k.

$\therefore$ $\dfrac{K(k)}{K(\sqrt{1-k^2})}$ is strictly increasing in k.

Hence $k = l$, and then $C_1 = 1$.

$$\therefore \quad F(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

※