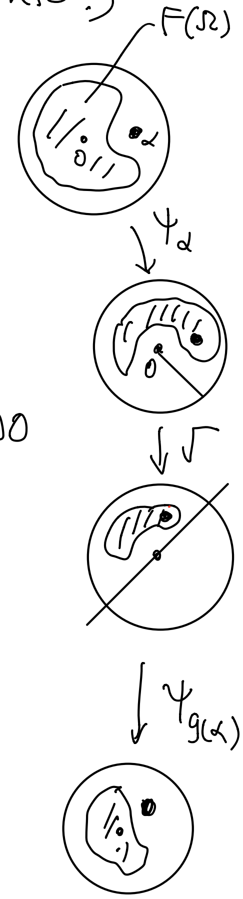
Step 4: The injective holo. map F found in Step 3 is conformal $F : \Omega \to \mathbb{D}$.

Pf: It remains to show: $F(\Omega) = \mathbb{D}$ (i.e. F is onto)

Suppose on the contrary that $\mathbb{D} \setminus F(\Omega) \neq \emptyset$.

i.e. $\exists\, \alpha \in \mathbb{D} \setminus \{0\}$ s.t. $F(z) \neq \alpha$, $\forall\, z \in \Omega$.

Then
$$\psi_\alpha \circ F(z) = \frac{\alpha - F(z)}{1 - \bar{\alpha} F(z)} \neq 0 \quad \forall\, z \in \Omega$$



Since Ω is simply-connected, $U = \psi_\alpha \circ F(\Omega)$ is also simply-connected as ψ_α & F are conformal to their respective images.

Hence
$$g : U \to \mathbb{C} : w \mapsto w^{\frac{1}{2}} = e^{\frac{1}{2}\log w}$$

can be defined.

Consider holo. $f = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F$.

Then $|\psi_\alpha \circ F| < 1 \Rightarrow |g \circ \psi_\alpha \circ F| = |\psi_\alpha \circ F|^{\frac{1}{2}} < 1$

$\therefore \quad f : \Omega \to \mathbb{D}$

Clearly, f is injective as square root g, ψ_α, $\psi_{g(\alpha)}$ and F are injective.

$$f(0) = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F(0)$$
$$= \psi_{g(\alpha)} \circ g \circ \psi_\alpha(0) = \psi_{g(\alpha)}\big(g(\alpha)\big) = 0$$

$\therefore \quad f \in \mathcal{F}$. $\qquad \left(\psi_{g(\alpha)}(z) = \dfrac{g(\alpha) - z}{1 - \overline{g(\alpha)}\, z} \right)$

Let $h(w) = w^2$, then

$$f = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ F \qquad (\psi_\alpha \circ \psi_\alpha = Id)$$

$$\Rightarrow \quad \psi_{g(\alpha)} \circ f = g(\psi_\alpha \circ F) \qquad \text{square root}$$

$$\Rightarrow \quad \psi_\alpha \circ F = h(\psi_{g(\alpha)} \circ f) \qquad \text{square}$$

$$\Rightarrow \quad F = (\psi_\alpha \circ h \circ \psi_{g(\alpha)}) \circ f.$$
$$= \Phi \circ f.$$

Note that $\Phi : \mathbb{D} \to \mathbb{D}$ holo. as $\psi_\alpha, \psi_{g(\alpha)}$ & $h : \mathbb{D} \to \mathbb{D}$ holo,

$$\Phi(0) = \psi_\alpha \circ h \circ \psi_{g(\alpha)}(0)$$
$$= \psi_\alpha \circ h(g(\alpha)) = \psi_\alpha(\alpha) = 0$$

Schwarz Lemma $\Rightarrow$

$$|\Phi'(0)| < 1 \qquad \text{as } \Phi \text{ is not a rotation.}$$

(otherwise $e^{i\theta} g(\alpha) = \Phi(g(\alpha)) = \psi_\alpha \circ h(0) = \psi_\alpha(0) = \alpha$

$\Rightarrow \quad |\alpha|^{\frac{1}{2}} = |\alpha| \Rightarrow |\alpha| = 1$ contradicts $\alpha \in \mathbb{D}$.)

as $\alpha \neq 0$ since $0 = F(0) \in F(\Omega)$

Hence

$$\sup_{f \in \mathcal{F}} |f'(0)| = s = |F'(0)| = |\Phi'(0)||f'(0)| < |f'(0)|$$

which is a contradiction. Hence $F(\Omega) = \mathbb{D}$.

<u>Final Step</u>: Choose $\theta \in \mathbb{R}$ suitably to conclude

$$\begin{cases} e^{i\theta} F(z) : \Omega \to \mathbb{D} \text{ is conformal,} \\ e^{i\theta} F(0) = 0 \\ (e^{i\theta} F)'(0) > 0 \end{cases}$$

#

§4 Conformal Mappings onto Polygons

"Explicit" formula of conformal mapping from $\mathbb{H}$ to polygons.

4.1 Some examples

Eg 1. Recall $f(z) = z^\alpha$ is a conformal map from $\mathbb{H}$ to the sector $\{z : 0 < \arg z < \alpha\pi\}$, $0<\alpha<2$
(Eg 2 of section 1, page 210 on the Textbook)

- Note that

$$z^\alpha = f(z) = \int_0^z f'(\zeta)\,d\zeta = \alpha \int_0^z \zeta^{\alpha-1}\,d\zeta$$

denote $\beta = 1-\alpha$, then

$$f(z) = z^\alpha = \alpha \int_0^z \zeta^{-\beta}\,d\zeta \quad \text{with } \alpha+\beta=1.$$

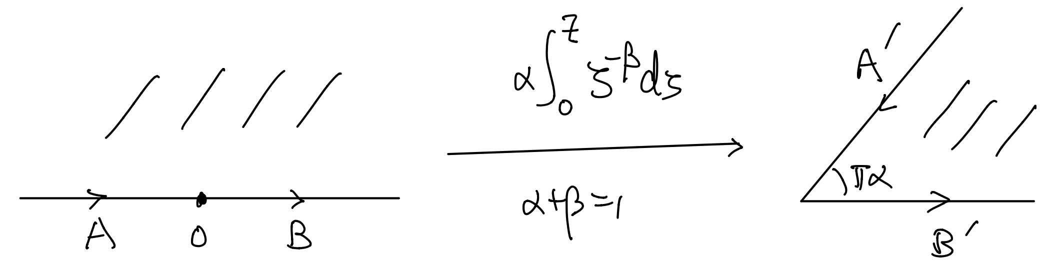
- The integral can be taken along any path in $\mathbb{H}$.
  Continuity $\Rightarrow$ any path in closure of $\mathbb{H}$,
  i.e. including line segments along the $\mathbb{R}$-axis.

- $0<\alpha<2 \Rightarrow \beta<1 \Rightarrow \zeta^{-\beta}$ integrable at $\zeta=0$.

$$\Rightarrow \begin{cases} f(z) = \int_0^z \zeta^{-\beta}\,d\zeta \text{ defined at } z=0 \text{ and} \\ f(0)=0 \end{cases}$$

(Original $z^\alpha = e^{\alpha \log z}$ is not defined for $z=0$!)

- Boundary mapping as in the figure



<u>Eg2</u> Consider $f: \mathbb{H} \to \mathbb{C}$

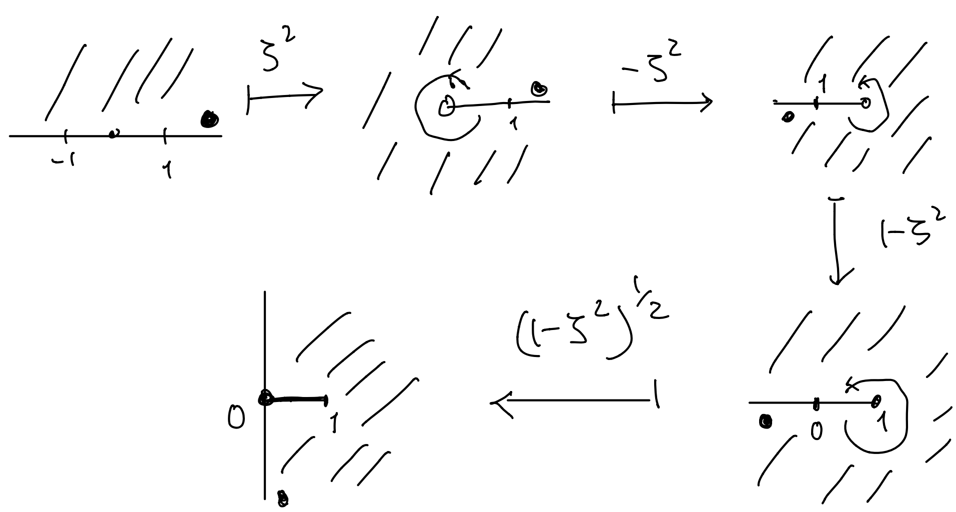
$$z \mapsto \int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}}$$

where integral taken along any path in closure($\mathbb{H}$),

with branch of square root s.t.

(i) $(1-\zeta^2)^{1/2}$ holo in $\mathbb{H}$;

(ii) $(1-\zeta^2)^{1/2} > 0$ for $-1<\zeta<1$.



- Singular points: $\zeta = \pm 1$ and

$$\int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \int_0^z \frac{d\zeta}{(1+\zeta)^{1/2}(1-\zeta)^{1/2}}$$

is integrable!

- For $z = x \in (-1,1)$,

take path = line segment from 0 to x on $\mathbb{R}$-axis,

$$\int_0^x \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \sin^{-1} x \quad \text{with principal branch } |\sin^{-1} x| < \frac{\pi}{2}$$

Taking limits, we see that

$$\int_0^{\pm 1} \frac{d\zeta}{(1-\zeta^2)^{1/2}} = \pm \frac{\pi}{2}$$

- For $\zeta > 1$,

$$\begin{cases} |(1-\zeta^2)^{1/2}| = (\zeta^2-1)^{1/2} \\ \arg (1-\zeta^2)^{1/2} = -\pi \end{cases}$$ according to

the choose of the branch (see figure above)

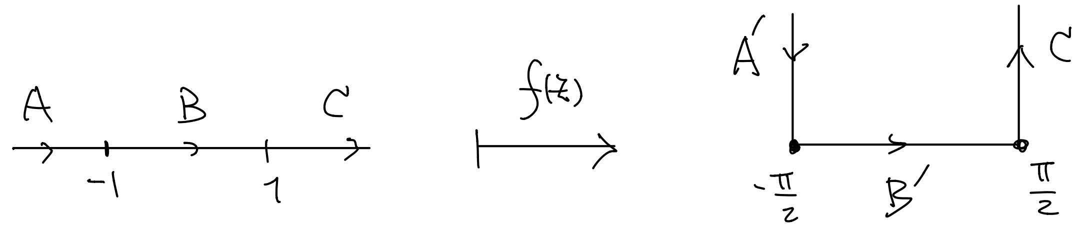
$\Rightarrow \quad (1-\zeta^2)^{1/2} = -i(\zeta^2-1)^{1/2}$

$\Rightarrow$ For $x > 1$,

$$\begin{aligned} f(x) = \int_0^x \frac{d\zeta}{(1-\zeta^2)^{1/2}} &= \int_0^1 \frac{d\zeta}{(1-\zeta^2)^{1/2}} + \int_1^x \frac{d\zeta}{(1-\zeta^2)^{1/2}} \\ &= \frac{\pi}{2} + \int_1^x \frac{d\zeta}{-i(\zeta^2-1)^{1/2}} \\ &= \frac{\pi}{2} + i \int_1^x \frac{d\zeta}{(\zeta^2-1)^{1/2}} \\ &= \frac{\pi}{2} + i \operatorname{ch}^{-1} x \qquad (\operatorname{ch} = \cosh) \end{aligned}$$

Similarly for $x < -1$. (Ex!)

Hence $f(z)$ maps the boundary $\mathbb{R}$-line to



- In fact, $f(z) = \sin^{-1} z$ (Ex!)
  (Refer to Eg 8 of section 1)

$\therefore$ f maps $\mathbb{H}$ conformally onto the half-infinite strip as in the figure.

<u>Eg 3</u> Consider

$$f(z) = \int_0^z \frac{d\zeta}{[(1-\zeta^2)(1-k^2\zeta^2)]^{1/2}}, \quad z \in \mathbb{H}$$

where
- $0 < k < 1$, k fixed
- the branch of $(1-\zeta^2)^{1/2}$ & $(1-k^2\zeta^2)^{1/2}$ is chosen s.t.
  - (i) holo. in $\mathbb{H}$;
  - (ii) real & positive for $-1 < \zeta < 1$ and $-\frac{1}{k} < \zeta < \frac{1}{k}$ respectively.

- $f(z)$ is an <u>elliptic integral</u> (related to calculating the arc-length of an ellipse).

- There are 4 poles along the $\mathbb{R}$-line

- Clearly integrable as the exponent is $\frac{1}{2}$.

- For $z = x$ with $-1 < x < 1$,

$$f'(x) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} > 0 \quad \text{(by the choice of branch)}$$

Together with $f'(-z) = f'(z)$, we have

$$f(\pm 1) = \pm \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

It is traditionally denote $K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

Then $f(\pm 1) = \pm K$ and

$f(x)$ increases from $-K$ to K

as x increases from -1 to 1.

- For $z = x$ with $1 < x < 1/k$.

Then along the path from 0 to x on the $\mathbb{R}$-line, we pass through the pole $z = 1$, and the choice of branching of the square root gives

$$\left[(1-\zeta^2)(1-k^2\zeta^2)\right]^{\frac{1}{2}} = -i\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}$$

(as in Eg 2)

Hence

$$f(x) = \int_0^x \frac{d\xi}{[(1-\xi^2)(1-k^2\xi^2)]^{1/2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_1^x \frac{dx}{-i\sqrt{(x^2-1)(1-k^2x^2)}}$$

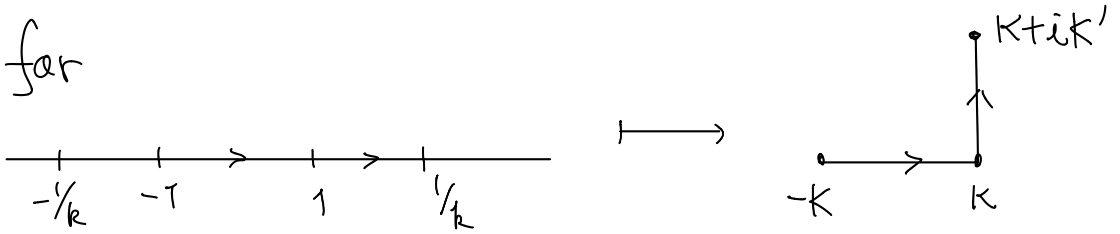
$$= K + i\int_1^x \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

$\therefore$ f maps the segment $(1, 1/k)$ to the vertical segment K to $K + iK'$,

where $$K' = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

with $f(1) = K$ to $f(1/k) = K + iK'$ as x goes from 1 to $1/k$.

So far

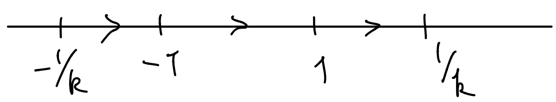


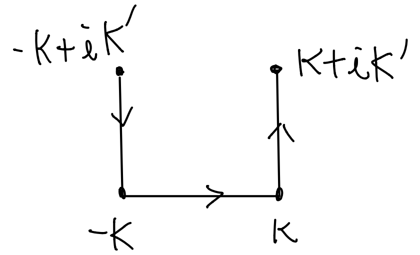
Similarly (Ex!), we have

$f([-\frac{1}{k}, -1])$ = vertical segment with end points $-K$ and $-K + iK'$

s.t. $f(-\frac{1}{k}) = -K + iK'$ to $f(-1) = -K$ as x goes from $-1/k$ to -1.

$\therefore$





- For $z = x$ with $x > 1/k$,

  we pass thro the pole $1/k$ too, therefore

$$[(1-\zeta^2)(1-k^2\zeta^2)]^{1/2} = -i\left(-i\sqrt{(x^2-1)(k^2x^2-1)}\right)$$
$$= -\sqrt{(x^2-1)(k^2x^2-1)}$$

$$\therefore \quad f'(x) = -\frac{1}{\sqrt{(x^2-1)(k^2x^2-1)}} < 0$$

And $$f(x) = K + iK' - \int_{1/k}^{x} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}}$$

$\therefore$ $f(x)$ belongs to the horizontal line $y = K'$

Note that $$\int_{1/k}^{x} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}} > 0$$

and $$\int_{1/k}^{\infty} \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}} = \int_{1}^{0} \frac{-\frac{1}{ku^2}\,du}{\sqrt{\left(\frac{1}{k^2u^2}-1\right)\left(\frac{1}{u^2}-1\right)}} \quad \left(x = \frac{1}{ku}\right)$$
$$= \int_{0}^{1} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = K$$

$\therefore$ f maps $(1/k, \infty)$ to the horizontal segment
$(iK', K+iK')$ (in reverse direction)

and

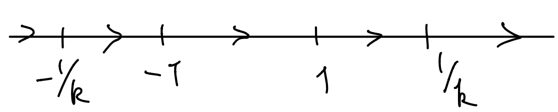
$$f\left(\tfrac{1}{k}\right) = K + iK', \quad \lim_{x\to+\infty} f(x) = iK'$$

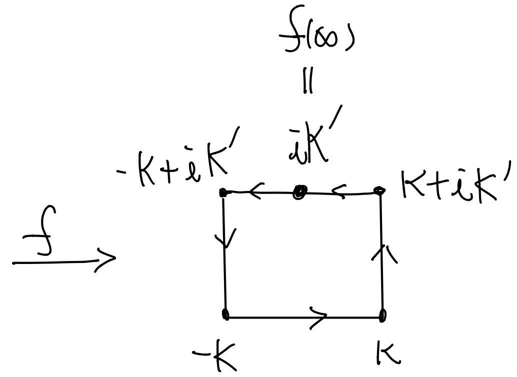
Similarly f maps $(-\infty, -1/k)$ to the horizontal segment
$(-K+iK', iK')$

and $f(-1/k) = -K + iK'$, $\lim_{x\to-\infty} f(x) = iK'$.

$\left(\text{In fact,} \quad \lim_{\substack{z\to\infty \\ z\in\mathbb{H}}} f(z) = iK'\right)$

So we have (on boundary)





(Of course, we haven't shown that $f(\mathbb{H})$ = interior of the rectangle in the figure, nor bijective yet)

4.2 The Schwarz-Christoffel Integral

Def Schwarz-Christoffel Integral:

$$S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}} \qquad (5)$$

where
- $A_1 < \cdots < A_n$ are n distinct points on the real axis;
- $\beta_k < 1$, $\forall\, k = 1, \cdots, n$ such that
$$1 < \sum_{k=1}^{n} \beta_k$$
- branch of $(x - A_k)^{\beta_k}$ is given as in Remark (ii) below

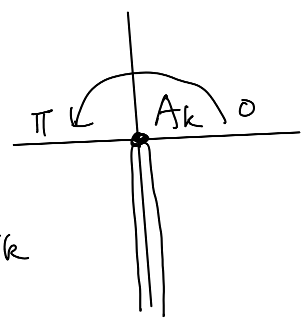
Remarks: (i) In Eg 1, $\beta = 1 - \alpha < 1$

Eg 2, $\beta_1 + \beta_2 = \frac{1}{2} + \frac{1}{2} = 1$

Eg 3, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2 > 1$.

In Egs 1 & 2, the image sets are not bounded polygons.

(ii) $(z - A_k)^{\beta_k}$ is the branch defined on
$$\mathbb{C} \setminus \{A_k + iy : y \le 0\}$$
s.t. $(x - A_k)^{\beta_k} > 0$ for $z = x > A_k$



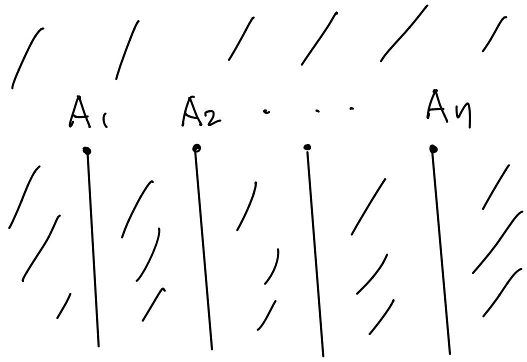
Then
$$(x - A_k)^{\beta_k} = \begin{cases} (x - A_k)^{\beta_k}, & \text{if } z = x > A_k \\ |x - A_k|^{\beta_k} e^{i\pi\beta_k}, & \text{if } z = x < A_k \end{cases}$$

(May be a different choose from the examples.)

(iii) Note that

$$\Omega = \mathbb{C} \setminus \bigcup_{k=1}^{n} \{A_k + iy : y \leq 0\}$$

is simply-connected,



so $S(z)$ is well-defined and holomorphic in Ω.

Moreover $\beta_k < 1 \Rightarrow \dfrac{1}{|\zeta - A_k|^{\beta_k}}$ is integrable near A_k.

(along any path in Ω to A_k)

$\therefore$ $S(z)$ extends continuously to the points A_k

with values $\boxed{S(A_k) = a_k}$, $k = 1, \cdots, n$

In particular,

$S(z)$ is continuous on $\mathbb{H} \cup \{\text{real-line}\}$

and holo. in $\mathbb{H}$.

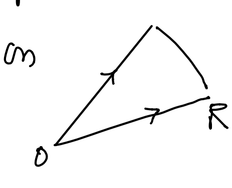
(iv)

$$\frac{1}{|(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}|} = \frac{1}{|\zeta - A_1|^{\beta_1} \cdots |\zeta - A_n|^{\beta_n}}$$

$$\leq \frac{1}{C\,|\zeta|^{\sum_{k=1}^{n} \beta_k}} \quad \text{for } |\zeta| \text{ large}$$

$\therefore$ $\sum_{k=1}^{n} \beta_k > 1 \Rightarrow$ The integral $S(z)$ converges at ∞.

$\Rightarrow$ $\boxed{\lim_{r \to \infty} S(re^{i\theta}) = a_\infty}$ exists and independent of θ, $0 < \theta < \pi$.

(Cauchy Thm on  & let $R \to \infty$)