Ch7 The Zeta Function and Prime Number Theorem

Def: The function $\pi(x)$ for $x>0$ is defined by

$$\pi(x) = \text{number of primes } p \leq x.$$

Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty$$

Recall: Asymptotic relation $f(x) \sim g(x)$ as $x \to \infty$ means that

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to \infty.$$

Goal of this Chapter: use $\zeta(s)$ to prove Prime Number Theorem.

1. Zeros of the Zeta Function

Relationship of $\zeta(s)$ to prime numbers:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \quad \text{Re}(s) > 1$$

where the infinite product is over all primes.

Pf: Fundamental theorem of Arithmetic $\Rightarrow$

$\forall n \in \{2, 3, \cdots\}$, $n = p_1^{k_1} \cdots p_m^{k_m}$ in a unique way

where p_i are primes & $k_i \geq 0$ are integers.

$\Rightarrow$ For integers $M > N$,

$$\prod_{p \le N} \left[1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] = \sum_{p_i \le N} \frac{1}{(p_1^{k_1} \cdots p_m^{k_m})^s}$$

with $k_i \le M$ and $m \le \pi(N)$

Note that $p_i \ge 2 \Rightarrow \forall\, n \le N$

$$n = p_1^{k_1} \cdots p_m^{k_m} \quad \text{for } p_i \le N \text{ and } k_i \le M$$

$$\therefore \prod_{p \le N} \left[1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] \ge \sum_{n=1}^{N} \frac{1}{n^s}$$

On the other hand, $\sum_{p_i \le N} \frac{1}{(p_1^{k_1} \cdots p_m^{k_m})^s}$ has only finitely many terms, we also have

$$\prod_{p \le N} \left[1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] \le \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) .$$

Note that we have used the uniqueness of prime factorization.

Now by $1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} = \frac{1 - (p^{-s})^{M+1}}{1 - p^{-s}}$

we have
$$\sum_{n=1}^{N} \frac{1}{n^s} \le \prod_{p \le N} \frac{1 - (p^{-s})^{M+1}}{1 - p^{-s}} \le \zeta(s)$$

Letting $M \to \infty$ $(M > N) \Rightarrow \sum_{n=1}^{N} \frac{1}{n^s} \le \prod_{p \le N} \frac{1}{1 - p^{-s}} \le \zeta(s)$.

Letting $N \to \infty$, we proved the Relation for $s > 1$. Then uniqueness of analytic continuation implies it holds for $\operatorname{Re} s > 1$. #

$\boxed{\text{Thm 1.1 The only zeros of } \zeta(s) \text{ outside the critical strip } 0 \le \mathrm{Re}(s) \le 1 \text{ are } -2, -4, -6, \cdots .}$

Pf: For $\mathrm{Re}(s) > 1$, $\zeta(s) = \prod_p \frac{1}{1-p^{-s}} > 0$.

For $\mathrm{Re}(s) < 0$, we use the functional equation

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s).$$

Rewrite the functional equation as

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \cdot \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

- $\mathrm{Re}(s) < 0 \Rightarrow \mathrm{Re}(1-s) > 1 \Rightarrow \zeta(1-s) \neq 0$
- clearly $\Gamma(\frac{1-s}{2}) \neq 0$ & $\pi^{s-\frac{1}{2}} \neq 0$
- and by Thm 1.6 $1/\Gamma(\frac{s}{2})$ has zeros at $\frac{s}{2} = 0, -1, -2, \cdots$

All together, the zeros of $\zeta(s)$ in $\mathrm{Re}(s) < 0$ are exactly $s = -2, -4, -6, \cdots$. ※

Remarks: (i) $\boxed{\text{Riemann hypothesis: The zeros of } \zeta(s) \text{ in the critical strip lie on the line } \mathrm{Re}(s) = \frac{1}{2}.}$

(ii) $s = -2, -4, -6, \cdots$ are called the trivial zeros of $\zeta(s)$.

$\boxed{\text{Thm 1.2} \quad \zeta(1+it) \neq 0, \quad \forall t}$

Remark: the pole $s=1$ (i.e. $t=0$) is included.

The proof needs some lemmas.

Lemma 1.3 If $\mathrm{Re}(s)>1$, then
$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$
for some $c_n \geq 0$.

Pf: For $s>1$,
$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log \frac{1}{1-p^{-s}}$$
$$= \sum_p \sum_{m=1}^{\infty} \frac{1}{m} (p^{-s})^m \qquad (\text{since } p^{-s} < p^{-1} < 1.)$$

Since the double sum converges absolutely, we have
$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm}.$$

Clearly, the absolute convergence of the double sum holds for $\mathrm{Re}(s)>1$ $\quad (|p^{-s}| = p^{-\mathrm{Re}\, s} < p^{-1} < 1)$, the RHS defines a holo. function on $\mathrm{Re}(s)>1$. Then uniqueness of analytic continuation $\Rightarrow$ $\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm} \quad \forall \mathrm{Re}(s)>1$.

Note that the general term of the sum is $\frac{1}{m}(p^m)^{-s}$, we have $\log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ with

$$c_n = \begin{cases} \frac{1}{m}, & \text{if } n = p^m \text{ for some prime } p \\ 0, & \text{otherwise.} \end{cases}$$

#

Lemma 1.4 $\forall \theta \in \mathbb{R}$, $3 + 4\cos\theta + \cos 2\theta \geq 0$

Pf: $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2$. #

Cor 1.5 If $s = \sigma + it$ with $\sigma > 1$ & $t \in \mathbb{R}$,

then $\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 0$

Pf: $\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|$

$= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|$

$= 3\operatorname{Re}[\log\zeta(\sigma)] + 4\operatorname{Re}[\log\zeta(\sigma+it)] + \operatorname{Re}[\log\zeta(\sigma+2it)]$

By Lemma 1.3

$= 3\sum_n c_n \operatorname{Re}(n^{-\sigma}) + 4\sum_n c_n \operatorname{Re}(n^{-(\sigma+it)}) + \sum_n c_n \operatorname{Re}(n^{-(\sigma+2it)})$

$= \sum_n c_n \left(3n^{-\sigma} + 4\operatorname{Re} e^{-(\sigma+it)\log n} + \operatorname{Re} e^{-(\sigma+2it)\log n}\right)$

$= \sum_n c_n [3n^{-\sigma} + 4n^{-\sigma}\cos(t\log n) + n^{-\sigma}\cos(2t\log n)]$

$= \sum_n c_n n^{-\sigma} [3 + 4\cos(t\log n) + \cos(2t\log n)]$

≥ 0 by Lemma 1.4 (& Lemma 1.3 that $c_n \geq 0$) #

Pf of Thm 1.2

Suppose on the contrary that

$$\zeta(1+it_0)=0 \quad \text{for some } t_0\neq 0.$$

We consider the 3 factors in Cor 1.5 for $\sigma\to 1$ & $t=t_0$.

Since $\zeta(s)$ is holo. near $s=1+it_0$, $t_0\neq 0$,

$$\zeta(s)=(s-(1+it_0))^m h(s) \quad \text{near } s=1+it_0$$

with
- $m\geq 1$
- $h(s)$ holo near $s=1+it_0$ and $h(1+it_0)\neq 0$.

Hence

$$(*)_1 \qquad |\zeta(\sigma+it_0)|^4 \leq C(\sigma-1)^4 \quad \text{as } \sigma\to 1.\ (\sigma>1)$$

for some const. $C>0$

Then using $s=1$ is a simple pole of $\zeta(s)$, we also have

$$(*)_2 \qquad |\zeta(\sigma)|^3 \leq \frac{C_1}{(\sigma-1)^3} \quad \text{as } \sigma\to 1 \quad (\sigma>1)$$

Finally, $\zeta(s)$ holo. near $s=1+2it_0$,

$$(*)_3 \qquad |\zeta(\sigma+2it_0)| \leq C_2 \quad \text{as } \sigma\to 1 \quad (\sigma>1)$$

Combining $(*)_1$, $(*)_2$, $(*)_3$ and Cor 1.5, we have

$$1\leq |\zeta^3(\sigma)\zeta^4(\sigma+it_0)\zeta(\sigma+2it_0)| \to 0 \quad \text{as } \sigma\to 1 \ (\sigma>1)$$

which is a contradiction. The proof is completed. ※

1.1 Estimates for $1/\zeta(s)$

Prop 1.6 $\forall \varepsilon>0$, $\exists C_\varepsilon>0$ s.t.

$$\frac{1}{|\zeta(s)|} \leq C_\varepsilon |t|^\varepsilon \quad \text{for } s=\sigma+it,\ \sigma\geq 1 \text{ and } |t|\geq 1.$$

Pf: By Cor 1.5 and $\zeta(s)$ only has a pole at $s=1$, we have

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|\geq 1, \quad \forall \sigma\geq 1$$

By Prop 2.7(i) of Ch6, (taking $\sigma_0=1$) $\qquad (C_1=C_1(\varepsilon)>0)$

$$|\zeta(\sigma+2it)|\leq C_1|t|^\varepsilon \quad \forall\ \sigma\geq 1\ \&\ |t|\geq 1.$$

Hence $\quad 1\leq |\zeta^3(\sigma)\zeta^4(\sigma+it)|\cdot C_1|t|^\varepsilon$

Then similar to $(*)_2$ in the proof of Thm 1.2,

$$|\zeta^3(\sigma)|\leq \frac{C_2}{(\sigma-1)^3} \quad \text{for } \sigma>1.$$

$(C_3=C_3(\varepsilon)>0)$

Hence $\quad |\zeta^4(\sigma+it)|\geq \dfrac{C_3(\sigma-1)^3}{|t|^\varepsilon} \quad \forall\ \sigma>1\ \&\ |t|\geq 1$

and clearly this inequality trivially holds for $\sigma=1$.

Hence $\qquad (C_4=C_4(\varepsilon)>0)$

(3) $\qquad |\zeta(\sigma+it)|\geq C_4(\sigma-1)^{\frac{3}{4}}|t|^{-\frac{\varepsilon}{4}}, \quad \forall\ \sigma\geq 1\ \&\ |t|\geq 1$

Note that by Prop 2.7 (ii) of Ch 6, we have

for $\sigma'>\sigma\geq 1$,

$$\begin{aligned}|\zeta(\sigma'+it)-\zeta(\sigma+it)| &\leq |\zeta'(\sigma_c+it)||\sigma'-\sigma| \quad \text{for some } \sigma\leq\sigma_c\leq\sigma' \\ &\leq C_5|t|^\varepsilon|\sigma'-\sigma| \qquad (C_5=C_5(\varepsilon)>0) \\ &\leq C_5|t|^\varepsilon(\sigma'-1). \qquad (\sigma'>\sigma\geq 1)\end{aligned}$$

Let $A = \left(\frac{C_4}{2C_5}\right)^4 > 0$.

<u>Case 1</u> $\sigma - 1 \geq A|t|^{-5\varepsilon}$

Then (3) $\Rightarrow$ $|\zeta(\sigma+it)| \geq C_4 (A|t|^{-5\varepsilon})^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}$

$$= (C_4 A^{\frac{3}{4}}) |t|^{-4\varepsilon}$$

<u>Case 2</u> $\sigma - 1 < A|t|^{-5\varepsilon}$

Take $\sigma' > \sigma$ such that $\sigma' - 1 = A|t|^{-5\varepsilon}$.

Then triangle inequality $\Rightarrow$

$$|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - |\zeta(\sigma'+it) - \zeta(\sigma+it)|$$

$$\geq C_4 (\sigma'-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon} (\sigma'-1)$$

$$= \left[C_4 (\sigma'-1)^{-\frac{1}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[C_4 \frac{1}{(A|t|^{-5\varepsilon})^{\frac{1}{4}}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[C_4 \cdot \frac{2C_5}{C_4} \cdot |t|^{\varepsilon} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= C_5 |t|^{\varepsilon} (\sigma'-1)$$

$$= C_5 A |t|^{-4\varepsilon}$$

Hence $\forall \varepsilon > 0$, $|\zeta(\sigma+it)| \geq C_\varepsilon |t|^{-4\varepsilon}$ where $C_\varepsilon = \min\{C_4 A^{\frac{3}{4}}, C_5 A\}$.

Replacing 4ε by ε, we have

$|\zeta(\sigma+it)| \geq C_\varepsilon |t|^{-\varepsilon}$ with a new C_ε. #

2. Reduction to the functions ψ and ψ_1

Def Tchebychev's ψ-function

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p$$

$$= \sum_{1 \le n \le x} \Lambda(n)$$

where $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ \& } m \ge 1 \\ 0, & \text{otherwise} \end{cases}$

Remarks: (i) The sum $\sum_{p^m \le x}$ is over those integers of the form p^m that are $\le x$.

(ii) $[u]$ = greatest integer $\le u$.

Prop 2.1 If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$

Pf omitted as it is completely a "real" analysis argument. (Reading Exercise)

Remark: Converse of Prop. 2.1 holds.

Def $\psi_1(x) = \int_1^x \psi(u)\,du$

Prop 2.2 If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$, and therefore $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

Pf omitted as it is completely a "real" analysis argument.
(Reading Exercise)

Prop 2.3 $\forall c > 1$

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \qquad (6)$$

(The integral is along the vertical line $\mathrm{Re}(s) = c$.)

Pf:

Step 1: $\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad \mathrm{Re}\, s > 1}$

In Lemma 1.3, we have proved that

$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-ms}$$

$$\Rightarrow \quad \frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{1}{m} (-m \log p)\, p^{-ms}$$

$$\Rightarrow \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} (\log p)\, p^{-ms} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

※

Step 2:

Lemma 2.4 If $c > 0$, then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(1+s)} ds = \begin{cases} 0, & \text{if } 0 < a \le 1 \\ 1 - \frac{1}{a}, & \text{if } 1 \le a \end{cases}$$

(The integral is along the vertical line $\mathrm{Re}(s) = c$.)

Clearly the integral converges as $|a^s| = a^c$.

Case 1 $\quad a \geq 1$.

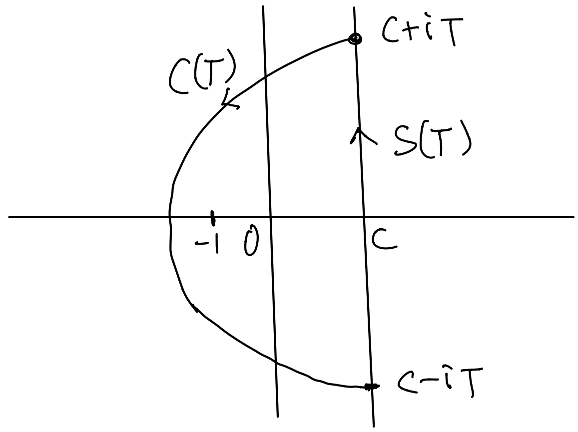
Let $\beta = \log a \geq 0$ and consider

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)} \qquad \text{which is meromorphic}$$

with simple poles at $s=0$ & $s=-1$ with

$$\operatorname{res}_{s=0} f = 1 \quad \text{and} \quad \operatorname{res}_{s=-1} f = -\frac{1}{a}$$

Let $\Gamma(T) = S(T) + C(T)$ be the contour as in the figure ($T > c+1$)



where $\quad S(T) =$ vertical line segment from $c-iT$ to $c+iT$;

$\quad C(T) =$ left half-circle of radius T centered at c.

Then Residue Theorem $\Rightarrow$

$$\frac{1}{2\pi i} \int_{\Gamma(T)} f(s)\, ds = 1 - \frac{1}{a}.$$

Now if $s = \sigma + it \in C(T)$, then $|s(s+1)| \geq (T-c)(T-c-1)$

$$\Rightarrow \quad \left| \int_{C(T)} f(s)\, ds \right| = \left| \int_{C(T)} \frac{e^{\beta s}}{s(s+1)}\, ds \right| \leq \int_{C(T)} \frac{|e^{\beta s}|}{|s(s+1)|}\, ds$$

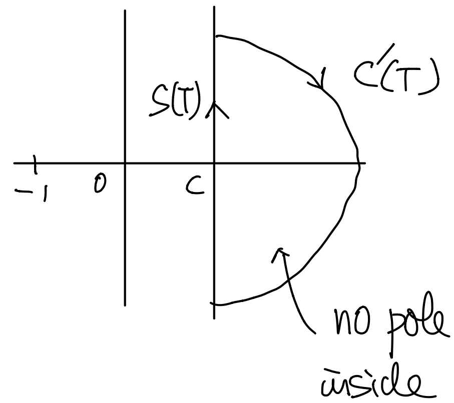
$$(\operatorname{Re} s \leq c,\ \beta \geq 0) \qquad \leq \frac{e^{\beta c}}{(T-c)(T-c-1)} \cdot \pi T \longrightarrow 0 \text{ as } T \to \infty,$$

$$\therefore \quad 2\pi i\left(1-\frac{1}{a}\right) = \int_{\Gamma(T)} f(s)\,ds = \int_{S(T)} f(s)\,ds + \int_{C(T)} f(s)\,ds$$

$$\to \int_{c-i\infty}^{c+i\infty} f(s)\,ds \quad \text{as } T \to \infty .$$

$$\therefore \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\,ds = 1-\frac{1}{a} \quad \text{for } a \geq 1 .$$

<u>Case 2</u> $0 < a \leq 1$



Similar to case 1, we have

$$\left|\int_{C'(T)} f(s)\,ds\right| \leq \int_{C'(T)} \frac{|a^s|}{|s(s+1)|}\,ds$$

$$= \int_{C'(T)} \frac{|e^{-s\log\frac{1}{a}}|}{|s(s+1)|}\,ds \qquad \left(\log\frac{1}{a} > 0\right)$$

$$(\operatorname{Re} s \geq c) \qquad \leq e^{-c\log\frac{1}{a}} \cdot \frac{1}{(T+c)(T+c+1)} \cdot \pi T \to 0$$

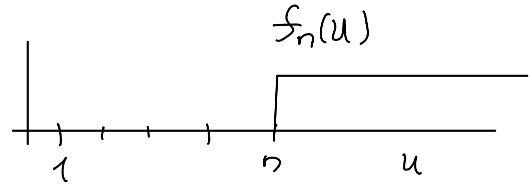
and the same argument gives

$$0 = \int_{S(T)} f(s)\,ds + \int_{C'(T)} f(s)\,ds$$

$$\to \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\,ds \quad \text{as } T \to \infty .$$

#

<u>Step 3</u> $\boxed{\psi_1(x) = \sum_{n \leq x} \Lambda(n)(x-n)}$



$$\psi(u) = \sum_{1 \leq n \leq u} \Lambda(n)$$

$$= \sum_{n=1}^{\infty} \Lambda(n) f_n(u)$$

where $f_n(u) = \begin{cases} 1 & \text{if } u \geq n \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \quad \psi_1(x) = \int_1^x \psi(u)\,du$$

$$= \int_0^x \psi(u)\,du \qquad \text{as } \psi(u)=0 \text{ for } 0<u\le 1$$

$$= \int_0^x \sum_{n=1}^{\infty} \Lambda(n) f_n(u)\,du$$

$$= \sum_{n=1}^{\infty} \Lambda(n) \int_0^x f_n(u)\,du$$

$$= \sum_{n\le x} \Lambda(n) \int_0^x f_n(u)\,du \qquad \text{as } n>x\ge u \Rightarrow f_n(u)=0$$

$$= \sum_{n\le x} \Lambda(n)(x-n) \qquad \#$$

Final Step: For $c>1$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

(by Step 1) $\operatorname{Re} s = c > 1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right) ds$$

$$= x\cdot\sum_{n=1}^{\infty}\Lambda(n)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\,ds$$

(by Step 2)

$$= x\sum_{n=1}^{\infty}\Lambda(n)\cdot\begin{cases} 1-\frac{1}{\left(\frac{x}{n}\right)}, & \text{if } \frac{x}{n}\ge 1 \\ 0, & \text{otherwise}\end{cases}$$

$$= x\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right)$$

$$= \sum_{n\le x}\Lambda(n)(x-n)$$

(by Step 3)

$$= \psi_1(x). \qquad \#$$