Pf of Thm 3.4

$\forall \varepsilon > 0$, consider $F_\varepsilon(z) = F(z) e^{-\varepsilon z^{\frac{3}{2}}}$ for $z \in S$.

Note that $z \in S \Rightarrow z = re^{i\theta}$ with $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

$$-\frac{3\pi}{8} < \frac{3\theta}{2} < \frac{3\pi}{8}$$

$\Rightarrow \quad \cos\frac{3\theta}{2} \geq \delta > 0$ for some δ.

Hence

$$|e^{-\varepsilon z^{\frac{3}{2}}}| = e^{-\varepsilon r^{\frac{3}{2}}\cos\left(\frac{3\theta}{2}\right)} \leq e^{-\varepsilon\delta r^{\frac{3}{2}}} \leq 1$$

Therefore, the growth condition on $F(z)$ implies

$$|F_\varepsilon(z)| = |F(z)||e^{-\varepsilon z^{\frac{3}{2}}}| \leq c_1 e^{c_2 r} e^{-\varepsilon\delta r^{\frac{3}{2}}}$$
$$= c_1 e^{-(\varepsilon\delta - c_2 r^{-\frac{1}{2}}) r^{3/2}}$$

For $r \gg 1$, $\varepsilon\delta - c_2 r^{-\frac{1}{2}} > 0$, hence

$F_\varepsilon(z)$ is rapidly decreasing.

In particular, F_ε is bounded on $\overline{S}$.

Let $M_\varepsilon = \sup_{z \in \overline{S}} |F_\varepsilon(z)|$.

If $F_\varepsilon \equiv 0$, then $F \equiv 0$, we are done.

If $F_\varepsilon \not\equiv 0$, then $\exists\ w_j \in S$, $j = 1, 2, \cdots$, s.t.

$$|F_\varepsilon(w_j)| \to M_\varepsilon \quad \text{as } j \to +\infty$$

and $M_\varepsilon > 0$.

As $|F_\varepsilon| \to 0$ as $|z| \to +\infty$, we conclude that $\{w_j\}$ is bounded. Therefore $\exists\ w \in \bar{S}$ s.t. $w_j \to w$. (by passing to subseq.)

By maximum principle (Thm 4.5), w can't be an interior point of S. Hence $w \in \partial S$.

Continuity of F on $\bar{S}$ and $|F| \leq 1$ on ∂S implies

$$M_\varepsilon = |F_\varepsilon(w)| \leq |F(w)|\,|e^{-\varepsilon w^{\frac{3}{2}}}| \leq 1.$$

i.e.

$$\left|F(z)\, e^{-\varepsilon z^{\frac{3}{2}}}\right| \leq 1, \quad \forall z \in S$$

$\Rightarrow$

$$|F(z)| \leq e^{\varepsilon |z|^{\frac{3}{2}}}, \quad \forall z \in S$$

Since $\varepsilon > 0$ is arbitary, $|F(z)| \leq 1$, $\forall z \in S$.

#

Ch5 Entire Functions

§1 Jensen's Formula

In this section, $D_R = \{z : |z| < R\}$ $(R > 0)$

$C_R = \{z : |z| = R\} = \partial D_R$

Thm 1.1 (Jensen's Formula)

Let
- Ω = open set s.t. $\overline{D_R} \subset \Omega$. (hence $0 \in \Omega$)
- f holo. on Ω,
- $f(z) \neq 0$ for $z = 0$ or $z \in C_R$
- $z_1, \cdots, z_N \in D_R$ are (all) the zeros of f in D_R (i.e. $z_1, \cdots, z_N \notin C_R$) (countable multiplicity)

Then

(1) $$\log|f(0)| = \sum_{k=1}^{N} \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

Pf: (My Steps are different from the Text)

Step 1 If g holo on $\overline{D_R}$ and $g(z) \neq 0$, $\forall z \in \overline{D_R}$, then

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta.$$

Pf: g holo on $\overline{D_R} \Rightarrow g$ holo on $D_{R+\varepsilon}$ for some $\varepsilon > 0$.

Since $D_{R+\varepsilon}$ is simply connected & $g(z) \neq 0$ on $D_{R+\varepsilon}$,

there exists a holo. function $h(z)$ on $D_{R+\varepsilon}$ s.t.

$(z) = e^{h(z)}$. (Thm 6.2 in Ch3 of Text)

$$\Rightarrow \quad |g(z)| = |e^{h(z)}| = e^{\operatorname{Re} h(z)}$$

By mean value property (of harmonic functions) (Cor 7.3 in Ch3 of Text),

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} h(Re^{i\theta})\,d\theta$$

$$= \operatorname{Re} h(0)$$

$$= \log|g(0)|. \quad \#$$

Step 2 $\quad \int_0^{2\pi} \log|1-ae^{i\theta}|\,d\theta = 0, \ \forall |a|<1.$

Pf: consider

$$F(z) = 1 - az \quad \text{on } \mathbb{D} = \{|z|<1\}$$

Then • $F(z)$ is holo. on $\overline{\mathbb{D}}$,

• $F(z) \neq 0$ on $\overline{\mathbb{D}}$, since $|a|<1$

By Step 1,

$$0 = \log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|1-ae^{i\theta}|\,d\theta \quad \#$$

Step3 General case.

Pf: By assumption & Thm 1.1 of Ch3,

$$f(z) = (z-z_1)\cdots(z-z_N)\,g(z) \quad \text{for some holo. function}$$

g on Ω s.t. $g(z) \neq 0, \ \forall z \in \overline{D_R}$.

Then $\log|f(0)| = \log|z_1\cdots z_N||g(0)|$

$$= \sum_{k=1}^{N} \log|z_k| + \log|g(0)|$$

(By Step 1) $$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta.$$

($z_k \notin C_R$) $$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{|f(Re^{i\theta})|}{|Re^{i\theta}-z_1|\cdots|Re^{i\theta}-z_N|}\,d\theta$$

$$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

$$- \sum_{k=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} \log R\left|1-\frac{z_k}{R}e^{-i\theta}\right|d\theta$$

(by change of variable in last term $\theta \mapsto -\theta$) $$= \sum_{k=1}^{N} \log\frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

$$- \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^{2\pi} \log\left|1-\frac{z_k}{R}e^{i\theta}\right|d\theta$$

($\frac{|z_k|}{R} < 1$ & Step 2) $$= \sum_{k=1}^{N} \log\frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

Def Notations as in Thm 1.1, we <u>define</u> the function of $r \in (0,R)$

$$n_f(r) = \text{number of zeros of } f \text{ in } D_r$$

(or simply $n(r)$) (counting multiplicity)

<u>Remark</u>: $r_1 > r_2 \Rightarrow n(r_1) \geq n(r_2)$ (nondecreasing)

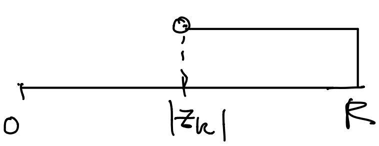
Lemma 1.2 If f holo on $\overline{D}_R$ & $f(0) \neq 0$.
If $z_1, \cdots, z_N$ are the zeros of f in D_R, then

$$\int_0^R n(r)\,\frac{dr}{r} = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|$$

<u>Pf</u>: Clearly $\displaystyle\sum_{k=1}^N \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r}$

Define the characteristic function

$$\eta_k(r) = \begin{cases} 1, & r > |z_k| \quad (r < R) \\ 0, & r \leq |z_k| \end{cases}$$



Then $$\sum_{k=1}^N \log\frac{R}{|z_k|} = \sum_{k=1}^N \int_0^R \eta_k(r)\,\frac{dr}{r}$$
$$= \int_0^R \left(\sum_{k=1}^N \eta_k(r)\right)\frac{dr}{r}$$

Note that $$\sum_{k=1}^N \eta_k(r) = \underbrace{1 + \cdots + 1}_{\text{those } k \text{ s.t. } r > |z_k|,\ (\text{i.e. } z_k \in D_r)} + 0 + \cdots + 0 = n(r)$$

we've proved the Lemma ※.

By the Lemma 1.2, the Jensen's formula can be rewritten as

$$\boxed{\begin{array}{c} (2) \qquad \displaystyle\int_0^R \pi(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)| \\ \\ \text{for } f \text{ holo. on } \overline{D}_R \text{ with } f(0)\neq 0 \;\&\; f(z)\neq 0 \;\forall z\in C_R \end{array}}$$