§2 Action of the Fourier Transform on $\mathfrak{F}$

<u>Def</u> Let $f: \mathbb{R} \to \mathbb{C}$. The <u>Fourier transform</u> of f is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \quad , \quad \xi \in \mathbb{R}$$

For $f \in \mathfrak{F}_a$, we consider the Fourier transform of $f(x)$, i.e. when $y=0$. Then we have

<u>Thm 2.1</u> If $f \in \mathfrak{F}_a$, for some $a>0$, then $\exists B>0$ s.t.

$$|\hat{f}(\xi)| \le B e^{-2\pi b|\xi|}, \quad \forall\ 0 \le b < a.$$

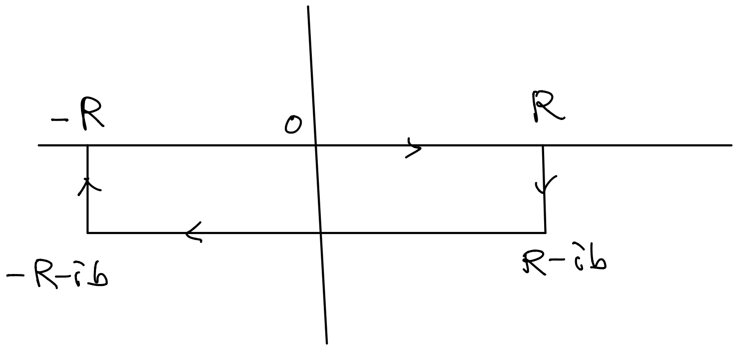
Pf: $|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \, dx$ since $x, \xi \in \mathbb{R}$

$\le \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx = B$ which is bounded,

$\Rightarrow \quad |\hat{f}(\xi)| \le B e^{-2\pi b|\xi|}$ holds for $b=0$.

For $0<b<a$:

If $\xi > 0$, consider the contour integral of the holo. function $g(z) = f(z) e^{-2\pi i z \xi}$ in S_a along the contour which is the boundary of the rectangle $[-R,R] \times [-b,0]$ $(R>0)$



On the vertical edge $[-R-ib, -R]$,
using parametrization $-R-it$, $t\in[0,b]$

$$\left|\int_{-R-ib}^{-R} g(z)\,dz\right| \leq \int_0^b \left|f(-R-it)\, e^{-2\pi i(-R-it)\xi}\right| dt$$

$$\leq \int_0^b \frac{A}{1+R^2}\, e^{-2\pi\xi t}\, dt \qquad \text{since } \xi>0$$

$$= \frac{A}{1+R^2}\int_0^b e^{-2\pi\xi t}\, dt \longrightarrow 0 \quad \text{as } R\to+\infty.$$

Similarly $\left|\int_R^{R-ib} g(z)\,dz\right| \leq \frac{A}{1+R^2}\int_0^b e^{-2\pi\xi t}\,dt \longrightarrow 0$ as $R\to+\infty$.

Therefore, Cauchy theorem $\Rightarrow$

$$\left|\int_{-R}^{R} f(x)e^{-2\pi i x\xi}\,d\xi - \int_{-R}^{R} f(x-ib)e^{-2\pi i(x-ib)\xi}\,dx\right| \leq \frac{2A}{1+R^2}\int_0^b e^{-2\pi\xi t}\,dt.$$

Letting $R\to+\infty$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,d\xi$$

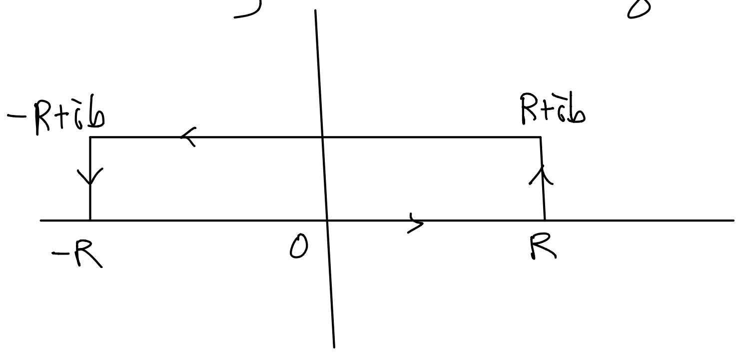
$$= \int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}\,dx \qquad (*),$$

$$\Rightarrow \quad |\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x-ib)|\, e^{-2\pi b\xi}\, dx \qquad (\xi>0)$$

$$\leq \left(\int_{-\infty}^{\infty} \frac{A}{1+x^2}\, dx\right) e^{-2\pi b\xi}$$

$$= B e^{-2\pi b|\xi|} \qquad (\xi>0)$$

For $\xi<0$, consider similarly the contour integral of $g(z)$ along :



$$\Rightarrow \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib)\, e^{-2\pi i(x+ib)\xi}\, dx \quad \text{———} \quad (*)_2 \quad (\text{Ex!})$$

and hence the result. $\#$

<u>Remark</u>: Therefore, if $f \in \mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$, then

$|\hat{f}(\xi)|$ decay exponentially as $|\xi| \to +\infty$,

in particular, it is <u>rapid decay at infinity</u> (i.e. decay faster than any $|\xi|^{-N}$, $\forall N>0$. More precisely $o(\frac{1}{|\xi|^N})$, $\forall N>0$.)

Thm 2.2 (Fourier Inversion Formula)

If $f \in \mathcal{F}$, then
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad \forall x \in \mathbb{R}$$

The proof needs a lemma:

Lemma 2.3 If $A > 0$ & $B \in \mathbb{R}$, then
$$\int_0^{\infty} e^{-(A+iB)\xi} d\xi = \frac{1}{A+iB}$$

Pf:

Note $A>0, B \in \mathbb{R} \Rightarrow |e^{-(A+iB)\xi}| = e^{-A\xi}$, for $\xi \in (0,\infty)$.

Hence the improper integral converges.

$$\therefore \int_0^{\infty} e^{-(A+iB)\xi} d\xi = \lim_{R \to +\infty} \int_0^R e^{-(A+iB)\xi} d\xi$$
$$= \lim_{R\to+\infty} \left[\frac{e^{-(A+iB)\xi}}{-(A+iB)} \right]_0^R = \frac{1}{A+iB}$$

\#

Pf of Thm 2.2 (Fourier Inversion Formula)

Note that $f \in \mathcal{F} \Rightarrow f \in \mathcal{F}_a$ for some $a>0$.

Then by equations $(*)_1$ & $(*)_2$ in the proof of Thm 2.1, (eqt (1) in the text)

If $\xi > 0$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i (x-ib)\xi} dx$, $\forall\, 0<b<a$.

If $\xi < 0$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib) e^{-2\pi i (x+ib)\xi} dx$, $\forall\, 0<b<a$

As sign of ξ is important in the proof, we write

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi + \int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

and work on the integrals individually:

$$\int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_{0}^{\infty} \left(\int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i (u-ib)\xi} du \right) e^{2\pi i x \xi} d\xi$$

Since $|f(u-ib)| \leq \frac{A}{1+u^2}$ (for some $A>0$), the (iterated) integrals are absolute convergence. Hence Fubini $\Rightarrow$

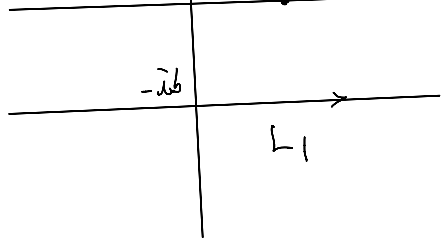
$$\int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_{-\infty}^{\infty} f(u-ib) \int_{0}^{\infty} e^{-2\pi i (u-ib)\xi} e^{2\pi i x \xi} d\xi \, du$$

$$= \int_{-\infty}^{\infty} f(u-ib) \left(\int_{0}^{\infty} e^{-2\pi i (u-x-ib)\xi} d\xi \right) du$$

$$(\text{Lemma } 2.3) = \int_{-\infty}^{\infty} f(u-ib) \frac{1}{2\pi b + 2\pi i (u-x)} du$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{(u-ib)-x} du$$

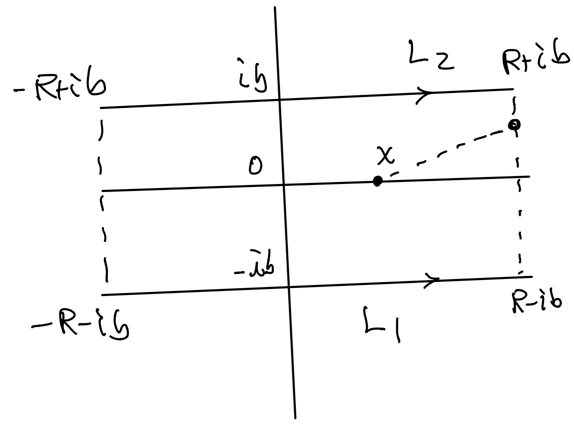
$$= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta$$



The contour integral of $\frac{f(z)}{z-x}$ (x fixed) along the horizontal line $y=-b$ (from left to right)

Similarly for $\xi < 0$,

$$\int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \frac{-1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta$$



where $L_2 : \{y = b\}$ from left to right.

Note that $\left| \int_{R-ib}^{R+ib} \frac{f(\zeta)}{\zeta - x} d\zeta \right| \leq 2b \frac{A}{1+R^2} \cdot \frac{1}{R-x}$ for $R > x$.

$\longrightarrow 0$ as $R \to +\infty$

Similarly,

$$\left| \int_{-R-ib}^{-R+ib} \frac{f(\zeta)}{\zeta - x} d\zeta \right| \to 0 \text{ as } R \to +\infty .$$

Cauchy integral formula, by letting $R \to +\infty$,

$$f(x) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta - \frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta$$

$$= \int_0^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi + \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi .$$

#

Thm 2.4 (Poisson Summation Formula)

If $f \in \mathfrak{F}$, then
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Pf: $f\in\mathfrak{F} \Rightarrow f\in\mathfrak{F}_a$ for some $a>0$.

$\Rightarrow f$ holo. on $S_a = \{x+iy : |y|<a\}$

Consider $g(z) = \dfrac{f(z)}{e^{2\pi i z}-1}$ on S_a.

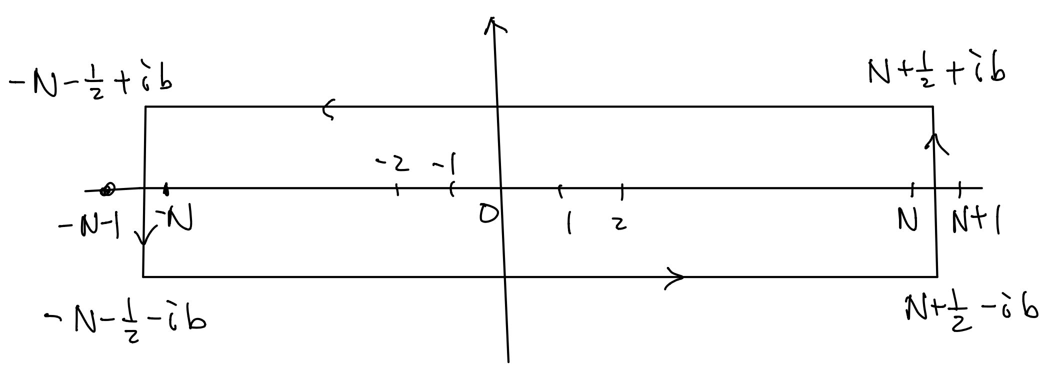
It is easy to see $\dfrac{1}{e^{2\pi i z}-1}$ has simple pole at $n\in\mathbb{Z}$

with $\quad \operatorname{res}_n \dfrac{1}{e^{2\pi i z}-1} = \dfrac{1}{2\pi i} \qquad$ (Ex!)

Hence $\quad g(z) = \dfrac{f(z)}{e^{2\pi i z}-1}$ has simple pole at $n\in\mathbb{Z}$

with $\quad \operatorname{res}_n g = \dfrac{f(n)}{2\pi i} \qquad$ (except $f(n)=0$, where g has a removable singularity $\Rightarrow$ no contribution to the contour integral.)

Applying Residue Formula (Cor 2.3 of Ch 3 of Text) to the contour γ_N, $N\in\mathbb{Z}^+$, as in the figure, for $0<b<a$,



we have

$$2\pi i \sum_{|n|\le N} \operatorname{res}_n g = \int_{\gamma_N} g(z)\,dz$$

i.e.

$$\sum_{|n|\le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz.$$

Note that $f \in \mathfrak{F}_a \Rightarrow \exists A>0$ s.t. $|f(z)| \le \dfrac{A}{1+|\mathrm{Re}(z)|^2}$

$$\Rightarrow |f(n)| \le \frac{A}{1+n^2} \quad \forall n \in \mathbb{Z}$$

$$\therefore \sum_{|n|\le N} f(n) \to \sum_{n\in\mathbb{Z}} f(n) \quad \text{as } N \to +\infty.$$

And

$$\left| \int_{\pm(N+\frac{1}{2})-ib}^{\pm(N+\frac{1}{2})+ib} \frac{f(z)}{e^{2\pi i z}-1}\,dz \right| \le \frac{C}{N^2} \quad (N \in \mathbb{Z}^+)$$

for some constant C depending on A and b only (Ex!)

Hence letting $N \to +\infty$ in $\displaystyle\sum_{|n|\le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz$,

we have

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z}-1}\,dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

where

$$\begin{cases} L_1 = \{x+iy : y=-b\} \text{ oriented left to right} \\ L_2 = \{x+iy : y=b\} \text{ oriented left to right.} \end{cases}$$

Note that on L_1, $|e^{2\pi i z}| = |e^{2\pi i (x-ib)}| = e^{2\pi b} > 1$

$$\therefore \quad \frac{1}{e^{2\pi i z}-1} = \frac{1}{e^{2\pi i z}} \cdot \frac{1}{1-e^{-2\pi i z}}$$

$$= e^{-2\pi i z} \sum_{k=0}^{\infty} e^{-2\pi i k z}$$

Similarly on L_2, $|e^{2\pi i z}| = e^{-2\pi b} < 1$

$$\frac{1}{e^{2\pi i z}-1} = -\sum_{k=0}^{\infty} e^{2\pi i k z}$$

$$\therefore \quad \sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} f(z)\, e^{-2\pi i z} \sum_{k=0}^{\infty} e^{-2\pi i k z}\, dz$$

$$+ \int_{L_2} f(z) \sum_{k=0}^{\infty} e^{2\pi i k z}\, dz$$

Since $|f(z)| \leq \frac{A}{1+|\operatorname{Re} z|^2}$, both $\int_{L_1}$ & $\int_{L_2}$ can be interchanged

with $\sum_{k=0}^{\infty}$, and we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k=0}^{\infty} \int_{L_1} f(z)\, e^{-2\pi i (k+1) z}\, dz$$

$$+ \sum_{k=0}^{\infty} \int_{L_2} f(z)\, e^{2\pi i k z}\, dz$$

Then using $(*)_1$ & $(*)_2$ in the proof of Thm 2.1, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k=0}^{\infty} \hat{f}(k+1) + \sum_{k=0}^{\infty} \hat{f}(-k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

#

Applications of Poisson summation formula

(1) For $t>0$, define the <u>theta function</u> by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

Then $\vartheta(t) = t^{-\frac{1}{2}}\vartheta(\frac{1}{t})$, $\forall\, t>0$.

<u>Pf</u>: This follows from a more general formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\pi\frac{n^2}{t}} e^{2\pi i n a} \quad \text{for } a\in\mathbb{R}.$$

To prove this, we observe that by Eg 1 of Ch 2,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx = e^{-\pi\xi^2}$$

(i.e. Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi\xi^2}$.)

Change of variable $x \mapsto \sqrt{t}(x+a) \Rightarrow$

$$\int_{-\infty}^{\infty} e^{-\pi t(x+a)^2} e^{-2\pi i x(\sqrt{t}\xi)} e^{-2\pi i a(\sqrt{t}\xi)}\sqrt{t}\,dx = e^{-\frac{\pi}{t}(\sqrt{t}\xi)^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi t(x+a)^2} e^{-2\pi i x\zeta}\,dx = \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}\zeta^2} e^{2\pi i a\zeta} \qquad (\zeta=\sqrt{t}\xi)$$

i.e. $$\hat{f}(\zeta) = \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}\zeta^2} e^{2\pi i a\zeta}$$

for the function $f(x) = e^{-\pi t(x+a)^2}$.

Then Poisson summation formula $\Rightarrow$ (Check $f\in\mathcal{S}$)

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}n^2} e^{2\pi i n a}$$

which is the required famula.

Putting $a=0$, we have

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right) \quad \#$$

(2) $\forall a \in \mathbb{R}$, $t>0$, we have

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh\left(\frac{\pi n}{t}\right)} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi(n+a)t)}$$

Pf: Eg 3 of Ch3 gives

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx = \frac{1}{\cosh(\pi\xi)}$$

Consider

$$f(x) = \frac{e^{-2\pi i a x}}{\cosh\left(\pi \frac{x}{t}\right)}, \text{ then}$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i a x}}{\cosh\left(\pi \frac{x}{t}\right)} e^{-2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-2\pi i a t x}}{\cosh(\pi x)} e^{-2\pi i t x \xi} t\, dx = \frac{t}{\cosh(\pi t(\xi+a))}$$

$\therefore$ Poisson summation famula $\Rightarrow$ (Check $f \in \mathfrak{F}$)

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh\left(\pi \frac{n}{t}\right)} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi t(n+a))} \quad \#$$

§3 Paley–Wiener Theorem

Omitted except the following theorem

Thm 3.4 (Phragmén–Lindelöf)

Suppose
- F is holo on $S = \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ and continuous on $\overline{S}$ (closure).
- $|F(z)| \le 1$ for $z \in \partial S$ (ie $|\arg z| = \frac{\pi}{4}$)

If $\exists$ constants $C_1, C_2 > 0$ such that

$$|F(z)| \le C_1 e^{C_2|z|}, \quad \forall z \in S,$$

then $|F(z)| \le 1$, $\forall z \in S$.

Remark: This is a "version" of maximum principle, but on unbounded domain.

$$\sup_{\overline{S}} |F(z)| = \sup_{\partial S} |F(z)|$$

which is usually not true without the growth condition.

eg
- $G(z) = e^{z^2}$ is holo on S
- $|G(re^{\pm i\frac{\pi}{4}})| = |e^{r^2 e^{\pm i\frac{\pi}{2}}}| = |e^{\pm r^2 i}| = 1$,

but $|G(x)| = e^{x^2} \to +\infty$ as $x \to +\infty$

$\therefore$ $G(z)$ is unbounded on S.