Def A region $\Omega \subset \mathbb{C}$ is simply connected if any two curves in Ω with the same end points are homotopic in Ω.

Thm 5.2 & Cor 5.3 If f is holomaphic in a simply connected domain Ω, then

(1) $\exists\ F : \Omega \to \mathbb{C}$ holo s.t. $F' = f$;

(2) $\int_\gamma f(z)\,dz = 0 \quad \forall$ closed curve γ in Ω.

§6 The Complex Logarithm

Thm 6.1 Suppose Ω is simply connected,

$1 \in \Omega$ but $0 \notin \Omega$.

Then $\exists$ a branch of the logarithm $F(z) = \log_\Omega z$ s.t.

(i) F is holo. in Ω

(ii) $e^{F(z)} = z, \ \forall z \in \Omega$

(iii) $F(r) = \log r \quad \forall r \in \mathbb{R}$ and near to 1.

- Principal branch of the logarithm

$$\begin{cases} \Omega = \mathbb{C} \setminus (-\infty, 0], \\ \log z = \log r + i\theta \quad \text{with } \boxed{|\theta| < \pi} \quad \text{and } z = re^{i\theta}. \end{cases}$$

Thm 6.2 Ω = <u>simply connected</u> region,
$f:\Omega\to\mathbb{C}$ holo. & $f(z)\neq 0, \forall z\in\Omega$.
Then $\exists$ $g:\Omega\to\mathbb{C}$ holo. s.t.
$$e^{g(z)} = f(z) \qquad \forall z\in\Omega.$$

(g is denoted by $\log f$)

§7 Fourier Series and Harmonic Functions

Thm 7.1 $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges in $D_R(z_0)$.
Then $\forall$ $r\in(0,R)$,
$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,e^{-in\theta}\,d\theta = \begin{cases} a_n r^n, & \text{for } n\geq 0 \\ 0, & \text{for } n<0 \end{cases}$$

<u>Remarks</u> (i) This is just the Cauchy integral formula applies to the circle
$$\gamma(\theta)=z_0+re^{i\theta}, \quad \theta\in[0,2\pi]$$

(ii) The LHS is the Fourier coefficients (up to a const.) of the 2π-periodic function $f(z_0+re^{i\theta})$ (for fixed r.)

Cor 7.2 & 7.3 $f=u+iv$ holo. in $D_R(z_0)$,
Then
$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta, \quad \forall\, 0<r<R.$$
$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta, \quad \forall\, 0<r<R$$

These are <u>mean-value property</u> for <u>holomorphic</u> and <u>harmonic</u> function respectively.

(End of Review)

Ch4 The Fourier Transform

§1 The Class $\mathcal{F}$

Def: $\forall a>0$, let $S_a = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < a\}$ (a horizontal strip)

Then

$$\mathcal{F}_a = \left\{ f : S_a \to \mathbb{C} : \begin{array}{l} f \text{ holo. on } S_a \text{ and } \exists A>0 \text{ s.t.} \\ |f(x+iy)| \le \frac{A}{1+x^2}, \ \forall x \in \mathbb{R} \ \& \ |y|<a \end{array} \right\}$$

and $\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$

Remark: For a fixed y, with $|y|<a$, the condition that

$$\exists A>0 \text{ s.t. } |f(x+iy)| \le \frac{A}{1+x^2}, \ \forall x \in \mathbb{R}$$

is usually referred as "moderate decay" on the horizontal line $\mathrm{Im}(z) = y$.

Hence, $f \in \mathcal{F}_a$ are moderate decay for each horizontal line $\mathrm{Im}(z)=y$, uniformly in $|y|<a$.

egs (i) Clearly $f(z) = e^{-\pi z^2} \in \mathcal{F}_a$, $\forall a>0$ (Ex!)

(ii) $\forall c>0$, the function

$$f(z) = \frac{1}{\pi} \frac{c}{c^2+z^2} \in \mathcal{F}_a, \ \forall a \in (0,c). \ (\text{Ex!})$$

Clearly, $f(z) \notin \mathcal{F}_a$ for $a \ge c$ as $z = \pm ci$ are poles.

Remarks: (1) For integer $n \geq 1$, $f \in \mathfrak{F}_a \Rightarrow f^{(n)} \in \mathfrak{F}_b$, $\forall\, 0<b<a$.
(Ex 2 of Ch 4 of Text)

(2) Many results in this chapter remain unchanged under the following weaker condition: $(\varepsilon > 0)$

$\exists A>0$ s.t. $|f(x+iy)| \leq \dfrac{A}{1+|x|^{1+\varepsilon}}$ $\forall\, x \in \mathbb{R}$ & $|y|<a$.

(Omitted)