MATH4060 Assignment 5

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1. (a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

   (b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3.$$

   Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j, j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Proof. (a) An automorphism of upper half plane must be of the form

$$\Phi(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{R}$. We now solves for its fixed points:

$$\frac{az+b}{cz+d} = z$$
$$cz^2 + (d-a)z - b = 0$$

We have a quadractic equation, so there are at most two roots, unless $c = b = 0, d = a$, in which case Φ is the identity function.

(b) Uniqueness follows from part (a). For the existence, it suffices to prove the special case $(x_1, x_2, x_3) = (-1, 0, 1)$. Then we consider the map

$$f(w) = \frac{w - y_2}{w - t} \cdot \frac{y_3 - t}{y_3 - y_2}$$

We have $f(y_2) = 0$, $f(y_3) = 1$. If $f(y_1) = -1$, then we can take $\Phi = f^{-1}$. So we need to solve for $t \in \mathbb{R}$ so that $f(y_1) = -1$:

$$\frac{y_1 - y_2}{y_1 - t} \cdot \frac{y_3 - t}{y_3 - y_2} = -1 \tag{1}$$

$$\frac{y_1 - y_2}{y_3 - y_2} = -\frac{y_1 - t}{y_3 - t} \tag{2}$$

It has a unique solution as it turns out to be a linear equation. However, we want f to preserve $\mathbb{H}$, that is

$$\frac{y_3 - t}{y_3 - y_2} \det \begin{pmatrix} 1 & -y_2 \\ 1 & -t \end{pmatrix} > 0$$

or, equivalently,

$$\frac{y_3 - t}{y_3 - y_2}(y_2 - t) > 0 \tag{3}$$

It can be seen from (2) that, for each case considered in the question, we do have $t < y_2$. For example, when $y_1 < y_2 < y_3$, we must have either $t < y_1$ or $t > y_3$, for each case it is clear that (3) is true.

□

2. Let

$$f(z) = \frac{i - z}{i + z} \quad \text{and} \quad f^{-1}(w) = i\frac{1 - w}{1 + w}$$

(a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$,

$$\frac{az + b}{cz + d} = f^{-1}(e^{i\theta} f(z)).$$

(b) Given $\alpha \in \mathbb{D}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$,

$$\frac{az + b}{cz + d} = f^{-1}(\psi_\alpha(f(z))).$$

(c) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathbb{H}$,

$$\frac{az + b}{cz + d} = f^{-1} \circ g \circ f(z).$$

Proof. (a)

$$\begin{aligned}
&\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \\
=&2ie^{i\theta/2} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}
\end{aligned}$$

so $a = d = \cos\theta/2, b = -c = \sin\theta/2$.

(b)

$$\begin{aligned}
&\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \\
=&2 \begin{pmatrix} \mathrm{Im}\alpha & -1 + \mathrm{Re}\alpha \\ 1 + \mathrm{Re}\alpha & -\mathrm{Im}\alpha \end{pmatrix}
\end{aligned}$$

so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{2}{1 - |\alpha|^2} \begin{pmatrix} \mathrm{Im}\alpha & -1 + \mathrm{Re}\alpha \\ 1 + \mathrm{Re}\alpha & -\mathrm{Im}\alpha \end{pmatrix}$$

(c) It is a combination of (a) and (b) because an automorphism of $\mathbb{D}$ is a composition of a rotation and some ψ_α, and because of the closedness of $\mathrm{SL}(\mathbb{R})$ under matrix composition.

□

3. We consider conformal mappings to triangles.

   (a) Show that
   $$\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}dz,$$
   with $0 < \beta_1, \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of $0, 1$, and ∞, and with angles $\alpha_1\pi, \alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

   (b) What happens when $\beta_1 + \beta_2 = 1$?

   (c) What happens when $\beta_1 + \beta_2 < 1$?

   (d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is $\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$.

Proof. (a) By proposition 4.1 (more accurately, by the proof of the proposition, because the integral is not the Schwarz-Christoffel integral, for the difference the signs of the denominators), the map sends the boundary of $\mathbb{H}$ (together with the point $\{\infty\}$) to the triangle described. Let T be the triangle together with the region enclosed. Note that $\mathbb{C} \setminus \partial T$ has two connected components. We know that $f(\mathbb{H})$ is open, and $f(\overline{\mathbb{H}})$(including ∞) is compact. So $f(\overline{\mathbb{H}}) \setminus \partial T$ is both closed and open in $\mathbb{C} \setminus \partial T$, and so must be one of the its connected components. But $f(\overline{\mathbb{H}})$ is compact, we thus have $f(\overline{\mathbb{H}}) = T$. Now, since $f(\mathbb{H})$ is open, we have $f(\mathbb{H}) = T \setminus \partial T$.

(b) It becomes an unbounded region bounded two parallel half ways and one line segment. (like a possibly rotated version of figure 4 in page. 233.)

(c) It becomes an unbounded region bounded by two non-parallel half ways and one line segment.

(d) Using the formula of exercise 7, chapter 6,
$$\begin{aligned}&\int_0^1 t^{-\beta_1}(1-t)^{-\beta_2}dt\\ =&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(1-\alpha_3)}\\ =&\frac{\sin\alpha_3\pi}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\end{aligned}$$
similarly, using the substitution $t = 1 - 1/z$ and $t = 1/(1-z)$, the lengths of the other two sides can be found to be
$$\frac{\sin\alpha_1\pi}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \text{ and } \frac{\sin\alpha_2\pi}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$$

□

4. If P is a simply connected region bounded by a polygon with vertices $a_1, \ldots, a_n$ and angles $\alpha_1\pi, \ldots, \alpha_n\pi$, and F is a conformal map of the disc

$\mathbb{D}$ to P, then there exist complex numbers $B_1, \ldots, B_n$ on the unit circle, and constants c_1 and c_2 so that

$$F(z) = c_1 \int_1^z \frac{d\zeta}{(\zeta - B_1)^{\beta_1} \cdots (\zeta - B_n)^{\beta_n}} + c_2$$

Proof. Applying the transformation

$$z \mapsto i\frac{1-z}{1+z}$$

to the Schwarz-Christoffel integral, we get

$$\begin{aligned} &\int_1^\zeta \frac{d(i\frac{1-\zeta}{1+\zeta})}{(i\frac{1-\zeta}{1+\zeta}) - A_1)^{\beta_1} \cdots (i\frac{1-\zeta}{1+\zeta}) - A_n)^{\beta_n}} \\ =&\int_1^\zeta \frac{-\frac{2i}{(1+\zeta)^2}d\zeta}{(i\frac{1-\zeta}{1+\zeta}) - A_1)^{\beta_1} \cdots (i\frac{1-\zeta}{1+\zeta}) - A_n)^{\beta_n}} \\ &\int_1^\zeta \frac{d\zeta}{(\zeta - B_1)^{\beta_1} \cdots (\zeta - B_n)^{\beta_n}} \end{aligned}$$

where in the last line we make use of the condition that $\beta_1 + \cdots \beta_n = 2$ $\square$

5. Let, for $0 < k < 1$,

$$K(k) = \int_0^1 \frac{dx}{((1-x^2)(1-k^2x^2))^{1/2}} \text{ and } K'(k) = \int_0^{1/k} \frac{dx}{((x^2-1)(1-k^2x^2))^{1/2}}$$

Show that if $\tilde{k}^2 = 1 - k^2$ and $\tilde{k} > 0$, then

$$K'(k) = K(\tilde{k}).$$

Proof. Let $x = (1 - \tilde{k}^2y^2)^{-1/2}$, then $dx = \tilde{k}^2y(1 - \tilde{k}^2y^2)^{-3/2}dy$, and

$$\begin{aligned} K'(k) &= \int_0^{1/k} \frac{dx}{((x^2-1)(1-k^2x^2))^{1/2}} \\ &= \int_0^1 \frac{\tilde{k}^2y(1-\tilde{k}^2y^2)^{-3/2}dy}{((\tilde{k}^2y^2(1-\tilde{k}^2y^2)^{-1})(\tilde{k}^2(1-y^2)(1-\tilde{k}^2y^2)^{-1}))^{1/2}} \\ &= \int_0^1 \frac{dy}{((1-y^2)(1-\tilde{k}^2y^2))^{-1/2}} \\ &= K(\tilde{k}) \end{aligned}$$

$\square$