MATH4060 Assignment 4

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1. (a) Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'(z_0) \neq 0$, then f preserves angles at z_0.

   (b) Conversely, prove the following: suppose $f : \Omega \to \mathbb{C}$ is a complex-valued function, that is real-differentiable at $z_0 \in \Omega$, and $J_f(z_0) \neq 0$. If f preserves angles at z_0, then f is holomorphic at z_0 with $f'(z_0) \neq 0$.

Proof.

(a) Let γ, η be curves with $\gamma(t_0) = \eta(t_0) = z_0$. Then from $(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0), (f \circ \eta)'(t_0) = f'(z_0)\eta'(t_0)$, we have

$$\frac{((f \circ \gamma)'(t_0), (f \circ \eta)'(t_0))}{|(f \circ \gamma)'(t_0)||(f \circ \eta)'(t_0)|} = \frac{(\gamma'(t_0), \eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|}.$$

Provided $\gamma'(t_0), \eta'(t_0), f'(z_0) \neq 0$.

(b) We need to show that f satisfies the Cauchy Riemann equations. Write $f = u + iv = u(x, y) + iv(x, y)$, assume $z_0 = 0$. And suppose

$$T = \frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let γ, η be curves whose tangent vector at 0 are represented by column vectors α, β. Then the assumptions says that, for $c_{\alpha\beta} = \frac{|T\alpha||T\beta|}{|\alpha||\beta|} > 0$

$$(T\alpha, T\beta) = c_{\alpha\beta}(\alpha, \beta)$$
$$\alpha^t T^t T\beta = c_{\alpha\beta}\alpha^t\beta$$

Putting $\alpha, \beta = (1, 0)^t, (0, 1)^t$ (four combinations), we see that

$$T^t T = \begin{pmatrix} a^2 + c^2 & 0 \\ 0 & b^2 + d^2 \end{pmatrix}.$$

Putting $\alpha = (1, 0)^t, \beta = (1, 1)^t$, we see that $a^2+c^2 = b^2+d^2$. Replace f with a positive multiple, we may thus assume

$$T^t T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, either using the forms of orthogonal matrices in $\mathbb{R}^2$, or by puting $a = \cos\theta, b = \sin\theta$ and solve it, we must have

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

or

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The former satisfies the Cauchy Riemann equations, but the latter does not, so we need to rule out the latter possibility, but the latter is a reflection, so does not preserved the orientation.

□

2. Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane.

Proof. The function is clearly holomorphic, and we just need to prove its bijectivity. First, we show that the image of f is contained in $\mathbb{H}$. For this, suppose $z = x + iy$ with $|z| < 1, y > 0$, then

$$\begin{aligned} 2\text{Im}(f(z)) &= i(\overline{f(z)} - f(z)) \\ &= y\Big(\frac{1}{|z|^2} - 1\Big) \\ &> 0. \end{aligned}$$

Next we show that f is injective, so suppose $\text{Im}(v), \text{Im}(v) > 0$ and $f(u) = f(v)$. Then

$$\begin{aligned} u + \frac{1}{u} &= v + \frac{1}{v} \\ (u - v)(u + v + 1) &= 0. \end{aligned}$$

Since $\text{Im}(u + v + 1) > 0$, we must have $u = v$.
Finally, we prove the surjectivity, suppose $w \in \mathbb{H}$, we have to solve for $f(z) = w$, i.e.

$$\begin{aligned} z + \frac{1}{z} &= -2w \\ z^2 + 2wz + 1 &= 0 \end{aligned}$$

Let $\alpha, \beta = \frac{1}{\alpha}$ be the two roots of the above equation in z, with $|\alpha| \leq 1$. We must have $\text{Im}(\alpha) > 0$ because

$$\begin{aligned} -2\text{Im}(w) &= \text{Im}(\alpha + \beta) \\ &= \text{Im}(\alpha + \frac{\overline{\alpha}}{|\alpha|^2}) \\ &= \text{Im}(\alpha)(1 - \frac{1}{|\alpha|^2}). \end{aligned}$$

□

3. Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

   (a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}$$

   This leads to two separate cases: either $0 < y \leq 1/2$ and $\theta = \pi/2$, or $1/2 \leq y < 1$ and $\theta = -\pi/2$. In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \qquad \text{and} \qquad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}.$$

   (b) Deduce that

$$\begin{aligned} \frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}\tilde{f}_0(\varphi)d\varphi \\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}dt \end{aligned}$$

   (c) Use a similar argument to prove the formula for the integral

$$\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta - \varphi)\tilde{f}_1(\varphi)d\varphi.$$

Proof.

(a) First,

$$\begin{aligned} re^{i\theta} &= G(iy) \\ &= \frac{i - e^{\pi i y}}{i + e^{\pi i y}} \\ &= \frac{-\cos \pi y + i(1 - \sin \pi y)}{\cos \pi y + i(1 + \sin \pi y)} \\ &= i\frac{2\cos \pi y}{\cos^2 \pi y + (1 + \sin \pi y)^2} \\ &= i\frac{\cos \pi y}{1 + \sin \pi y} \end{aligned}$$

Then we have

$$\begin{aligned} r^2 &= \frac{\cos^2 \pi y}{(1 + \sin \pi y)^2} \\ &= \frac{(1 - \sin \pi y)(1 + \sin \pi y)}{(1 + \sin \pi y)^2} \\ &= \frac{1 - \sin \pi y}{1 + \sin \pi y} \end{aligned}$$

and

$$\begin{aligned}
P_r(\theta-\varphi) &= \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} \\
&= \frac{1-r^2}{1-2r\sin\theta\sin\varphi+r^2} \\
&= \frac{1-\frac{1-\sin\pi y}{1+\sin\pi y}}{1-2\frac{\cos\pi y}{1+\sin\pi y}\sin\varphi+\frac{1-\sin\pi y}{1+\sin\pi y}} \\
&= \frac{\sin\pi y}{1-\cos\pi y\sin\varphi}.
\end{aligned}$$

(b) From

$$\begin{aligned}
e^{i\varphi} &= \frac{i-e^{\pi t}}{i+e^{\pi t}} \\
&= \frac{1-e^{2\pi t}+2ie^{\pi t}}{1+e^{2\pi t}} \\
&= -\tanh\pi t+i\frac{1}{\cosh\pi t},
\end{aligned}$$

we see that

$$\sin\varphi=\frac{1}{\cosh\pi t},\cos\varphi=-\tanh\pi t,$$

and

$$\begin{aligned}
\cos\varphi\frac{d\varphi}{dt} &= -\frac{\pi\tanh\pi t}{\cosh\pi t} \\
\frac{d\varphi}{dt} &= \frac{\pi}{\cosh\pi t}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi}\int_0^\pi P_r(\theta-\varphi)\tilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\tilde{f}_0(\varphi)d\varphi \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\sin\pi y}{1-\cos\pi y\frac{1}{\cosh\pi t}}f_0(t)\frac{\pi}{\cosh\pi t}dt \\
&= \frac{\sin\pi y}{2}\int_{-\infty}^\infty \frac{f_0(t)}{\cosh\pi t-\cos\pi y}dt
\end{aligned}$$

(c) For

$$\begin{aligned}
e^{i\varphi} &= \frac{i+e^{\pi t}}{i-e^{\pi t}} \\
&= \frac{1-e^{2\pi t}-2ie^{\pi t}}{1+e^{2\pi t}} \\
&= -\tanh\pi t-i\frac{1}{\cosh\pi t},
\end{aligned}$$

we see that

$$\sin\varphi=-\frac{1}{\cosh\pi t},\cos\varphi=-\tanh\pi t,$$

and

$$\begin{aligned}\cos\varphi\frac{d\varphi}{dt} &= \frac{\pi\tanh\pi t}{\cosh\pi t}\\ \frac{d\varphi}{dt} &= -\frac{\pi}{\cosh\pi t}\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta-\varphi)\tilde{f}_1(\varphi)d\varphi &= \frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\tilde{f}_1(\varphi)d\varphi\\ &= \frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{\sin\pi y}{1-\cos\pi y\frac{-1}{\cosh\pi t}}f_1(t)\frac{-\pi}{\cosh\pi t}dt\\ &= \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_1(t)}{\cosh\pi t+\cos\pi y}dt\end{aligned}$$

□

4. Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \le M$ for some $M > 0$, then

$$\left|\frac{f(z)-f(0)}{M^2-\overline{f(0)}f(z)}\right| \le \frac{|z|}{MR}.$$

Proof. For this question, we need to assume f is not a constant. Then since f has no maximal, we see that f is a function

$$f : D(0,R) \to D(0,M)$$

Consider the function

$$g : \mathbb{D} \to \mathbb{D}$$

defined by

$$g(w) = \frac{\frac{f(Rw)}{M}-\frac{f(0)}{M}}{1-\overline{\frac{f(0)}{M}}\frac{f(Rw)}{M}} = M\frac{f(Rw)-f(0)}{M^2-\overline{f(0)}f(Rw)}$$

By Schwarz lemma, we have $|g(w)| \le |w|$, putting $z = Rw$, we get the desired result.

□

5. Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}, \theta \in \mathbb{R}, \beta \in \mathbb{H}.$$

Proof. Recall that we have a conformal mapping

$$\phi : \mathbb{H} \to \mathbb{D}$$

given by

$$\phi(z) = \frac{z-i}{z+i}.$$

Let $f : \mathbb{H} \to \mathbb{D}$ be another conformal mappping, so $f \circ \phi^{-1}$ is an automorphism of $\mathbb{D}$. We know that $f \circ \phi^{-1}$ must be of the form

$$z \mapsto e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}$$

with $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$. So we have

$$\begin{aligned} f(z) &= e^{i\theta} \frac{\frac{z-i}{z+i} - \alpha}{1 - \overline{\alpha} \frac{z-i}{z+i}} \\ &= e^{i\theta} \frac{(1-\alpha) - i(1+\alpha)}{(1-\overline{\alpha}) + i(1+\overline{\alpha})} \\ &= e^{i\theta} \frac{z - \beta}{z - \overline{\beta}} \end{aligned}$$

where $\beta = i \frac{1+\alpha}{1-\alpha} = \phi^{-1}(\alpha) \in \mathbb{H}$. □