MATH4060 Assignment 1

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1. Show that for $a \in \mathbb{Z}$ and $t > 0$ the following functions belong to class $\mathcal{F}$.
   (a) $f(x) = e^{-\pi t(x+a)^2}$.
   (b) $f(x) = \frac{e^{-2\pi i a x}}{\cosh(\frac{\pi x}{t})}$.

   Proof.

   (a) We prove $f \in \mathcal{F}_1$, so let $z = x + iy$ with $|y| < 1$,

   $$\begin{aligned} |f(z)| &= e^{\pi t y^2} e^{-\pi t(x+a)^2} \\ &\leq e^{\pi t} e^{-\pi t(x+a)^2} \end{aligned}$$

   We make a general remark that if a continuous function $g : \mathbb{R} \to \mathbb{R}$ has a limit at infinity, then it must be bounded. We want to show that $(1 + x^2)e^{-\pi t(x+a)^2}$ is bounded, so we calculate its limit, using L'Hôpital's rule,

   $$\begin{aligned} \lim_{x\to\infty} (1 + x^2)e^{-\pi t(x+a)^2} &= \lim_{x\to\infty} \frac{(1 + x^2)}{e^{\pi t(x+a)^2}} \\ &= \lim_{x\to\infty} \frac{2x}{2\pi t(x + a)e^{\pi t(x+a)^2}} \\ &= \lim_{x\to\infty} \frac{1}{\pi t(1 + \frac{a}{x})e^{\pi t(x+a)^2}} \\ &= 0. \end{aligned}$$

   So we are done in showing $f \in \mathcal{F}_1$.

   (b) The idea is similar to a).
   We prove $f \in \mathcal{F}_{\frac{\pi t}{4}}$, so let $z = x + iy$ with $|y| < \frac{\pi t}{4}$. The norm of the numerator of f:

   $$e^{2\pi a y} \leq e^{\frac{\pi^2 a t}{2}}$$

   is bounded. On the other hand, the norm square of the denominator is [1]:

$$
\begin{aligned}
|\cosh(\frac{\pi z}{t})|^2 &= \cosh^2(\frac{\pi x}{t})\cos^2(\frac{\pi y}{t}) + \sinh^2(\frac{\pi x}{t})\sin^2(\frac{\pi y}{t}) \\
&\geq \cosh^2(\frac{\pi x}{t})\cos^2(\frac{\pi}{4}) \\
&= \frac{\cosh^2}{2}.
\end{aligned}
$$

It remains to calculate the limit:

$$
\begin{aligned}
\lim_{x\to\infty} \frac{1+x^2}{\cosh(\frac{\pi x}{t})} &= \lim_{x\to\infty} \frac{2tx}{\pi\sinh(\frac{\pi x}{t})} \\
&= \lim_{x\to\infty} \frac{2t^2}{\pi^2\cosh(\frac{\pi x}{t})} \\
&= 0.
\end{aligned}
$$

□

2. If $f \in \mathcal{F}_a, a > 0$. Then for any positive integer n, $f^{(n)} \in \mathcal{F}_b$ whenever $0 < b < a$.

   Proof. Let $\delta = a - b > 0$, then for any $z = x + iy \in S_b$, then disc $D_\delta(z)$ centered at z with radius δ lies inside S_a. Cauchy estimate says that

   $$|f^{(n)}(z)| \leq \frac{n!||f||_{D_\delta(z)}}{\delta^n}.$$

   Let A be the constant associated with the definition of $f \in \mathcal{F}_a$. Then for any $z' = x' + iy' \in D_\delta(z)$, we have (for x large)

   $$|f(z')| \leq \frac{A}{1+x'^2} \leq \frac{A}{1+(|x|-\delta)^2}.$$

   Finally, note that since

   $$\lim_{x\to\infty} \frac{1+x^2}{1+(|x|-\delta)^2} = 1,$$

   there exists a constant C such that $1+(|x|-\delta)^2 \geq C(1+x^2)$ for all $x \in \mathbb{R}$. Combining the above, we have

   $$|f^{(n)}(z)| \leq \frac{C'}{1+x^2}$$

   with

   $$C' = \frac{n!A}{\delta^n C}.$$

   □

[1]Since $\cosh(ix) = \cos(x), \sinh(ix) = i\sin(x)$, we have $\cosh(x+iy) = \cos(y-ix) = \cos y \cosh x + i\sin(y)\sinh(x)$

3. Suppose Q is a polynomial of deg ≥ 2 with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)} dx$$

in terms of the roots of Q. What happens when several roots coincide?

Proof. Suppose $Q(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_n$, $(a_0 \neq 0)$ we use the following lower bound of $|Q(z)|$ for $R = |z|$ large:

$$\begin{aligned}|Q(z)| &\geq |a_0||z|^n - |a_1||z|^{n-1} - \cdots |a_n| \\ &= R^n(|a_0| - \frac{|a_1|}{R} - \cdots \frac{|a_n|}{R^n}) \\ &\geq CR^n \\ &\geq CR^2\end{aligned}$$

for some constant C.

Now, we assume $\xi \leq 0$. Choose an arbitrarily large $R \in \mathbb{R}$ so that all roots of Q has modulus less than R. Let C_R be the upper half circle of radius R centered at the origin (running in anti-clockwise direction), by residue theorem, we have

$$\int_{C_R} \frac{e^{-2\pi i x z}}{Q(z)} dz + \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{Q(x)} dx = 2\pi i \sum_{\alpha_i} \frac{e^{-2\pi i \alpha_i \xi}}{Q'(\alpha_i)} \tag{1}$$

Where the sum on the right hand side is over all the roots of $Q(z)$ lying in the upper half-plane.
For the first term on the left hand side, we have

$$\begin{aligned}\left| \int_{C_R} \frac{e^{-2\pi i z\xi}}{Q(z)} dz \right| &\leq \int_0^{\pi} \frac{e^{2\pi R\xi \sin\theta}}{CR^2} R d\theta \\ &\leq \int_0^{\pi} \frac{1}{CR} d\theta \\ &= \frac{\pi}{CR}.\end{aligned}$$

Therefore, if we take $R \to \infty$, we see that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)} dx = 2\pi i \sum_{\alpha_i} \frac{e^{-2\pi i \alpha_i \xi}}{Q'(\alpha_i)}$$

the sum over all the roots of $Q(z)$ lying in the upper half-plane.
Consider the polynomial $\tilde{Q}(x) = Q(-x)$ and using the substitution $x \mapsto -x$, we have for $\xi \geq 0$ that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)} dx = -2\pi i \sum_{\beta_j} \frac{e^{-2\pi i \beta_j \xi}}{Q'(\beta_j)}$$

the sum over all the roots of $Q(z)$ lying in the lower half-plane.
For multiple roots, the idea is the same, but the formula for the residues would be more complicated. □

4. Prove that
$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2}=\sum_{n=-\infty}^{\infty}e^{-2\pi a|n|}$$
whenever $a>0$. Hence show that the sum equals $\coth(\pi a)$.

Proof. This would be the Poisson Summation formula. In fact let $f(x)=\frac{a}{\pi(a^2+x^2)}$, $g(x)=e^{-2\pi a|x|}$ (both of them are of class $\mathcal{F}$), then the formula reads
$$\sum_{-\infty}^{\infty}f(n)=\sum_{-\infty}^{\infty}g(n).$$
It remains to relate f and g using Fourier transform. It will be easier to calculate $\hat{g}$. (You need Contour Integral to calculate $\hat{f}$)
$$\begin{aligned}\hat{g}(\xi)&=\int_{-\infty}^{\infty}e^{-2\pi a|x|}e^{-2\pi ix\xi}dx\\&=\int_{-\infty}^{0}e^{2\pi(a-i\xi)x}dx+\int_{0}^{\infty}e^{-2\pi(a+i\xi)x}dx\\&=\left.\frac{e^{2\pi(a-i\xi)x}}{2\pi(a-i\xi)}\right|_{x=-\infty}^{x=0}+\left.\frac{e^{-2\pi(a+i\xi)x}}{-2\pi(a+i\xi)}\right|_{x=0}^{x=\infty}\\&=\frac{1}{2\pi(a-i\xi)}+\frac{1}{2\pi(a+i\xi)}\\&=\frac{a}{\pi(a^2+\xi^2)}\\&=f(\xi).\end{aligned}$$
For the last part, we do the calculations:
$$\begin{aligned}\sum_{n=-\infty}^{\infty}e^{-2\pi a|n|}&=-1+2\sum_{n=0}^{\infty}e^{-2\pi a|n|}\\&=-1+\frac{2}{1-e^{-2\pi a}}\\&=\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}\\&=\frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}}\\&=\coth(\pi a).\end{aligned}$$
□

5. (a) Let F be a holomorphic function in the right half-plane that extends continuously to the imaginary axis. Suppose $|F|\le 1$ on the imaginary axis, and
$$|F(z)|\le Ce^{c|z|^\gamma}$$
for some $C,c>0$ and $\gamma<1$. Prove that $|F|\le 1$ on the right half-plane.

(b) More generally, let S be a sector whose vertex is the origin, and forming an angle of π/β. Let F be a holomorphic function in S that is continuous on the boundary. Suppose $|F| \le 1$ on the boundary, and
$$|F(z)| \le Ce^{c|z|^\alpha}$$
for some $C, c > 0$ and $0 < \gamma < \beta$. Prove that $|F| \le 1$ on S.

Proof.

(a) The case $\gamma \le 0$ would be easy, and the case $\gamma > 0$ would follows from part b). So we prove part b) only. (In fact $\gamma \le 0$ also works for part b).

(b) Fix α with $\gamma < \alpha < \beta$. By using a rotation, we may assume S is the set
$$\{z \in \mathbb{C} : -\pi/2\beta < \arg(z) < \pi/2\beta\}.$$
For any small positive ϵ, Let $G_\epsilon(z) = F(z)e^{-\epsilon z^\alpha} = F(z)e^{-\epsilon \exp(\alpha \log(z))}$, note that we can choose a well-defined and holomorphic branch of log on S. For any $z = Re^{i\theta} \in S$,
$$\mathrm{Re}(z^\alpha) = R^\alpha \cos(\alpha\theta) \ge \delta R^\alpha,$$
where
$$\delta = \cos(\frac{\pi\alpha}{\beta}) > 0.$$

Therefore,
$$\begin{aligned} |G_\epsilon(z)| &\le |F(z)|e^{-\epsilon\delta R^\alpha} \\ &= \left(|F(z)|e^{-cR^\gamma}\right)e^{R^\gamma(c-\epsilon\delta R^{\alpha-\gamma})} \end{aligned}$$

The term in the parenthesis is bounded by assumption, and the remaining term vanishes at the infinity since $\alpha > \gamma$. This shows that G_ϵ vanished at the infinity, and so maximal $M = \max_{\overline{S}} G_\epsilon$ must achieved at some point a. If $a \in S$, the maximal modulus principle implies that $F \cong 0$. So if $F \ncong 0$, a must be on the boundary, while $|G_\epsilon| \le 1$ on the boundary, we thus have
$$|G_\epsilon| \le 1.$$
Taking $\epsilon \to 0^+$, we get the desired result. □