Mathematical Analysis III

Tutorial 3 (October 3)

The following were discussed in the tutorial this week:

1. (Question 2 in HW 2) Let f be a 2π-periodic function integrable on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} f = 0$. Define F by

$$F(x) = \int_0^x f(y)dy.$$

Then F is a 2π-periodic continuous function.

   (a) We show that $c_n(F) = \frac{1}{in}c_n(f)$ for all $n \neq 0$ via approximating integrable functions by continuous functions (under integral).

   (b) Suppose further that f satisfies a Lipschitz condition. We show that $c_0(F) = \sum_{n=1}^{\infty} b_n(f)/n$.

2. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{C}$. Write $\|f\|_2 := \sqrt{\langle f, f\rangle}$. Suppose $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of orthonormal set in X, that is

$$\langle \phi_n, \phi_m\rangle = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

For any $f \in X$, define

$$S_N(f) = \sum_{n=1}^{N} \langle f, \phi_n\rangle \phi_n \quad \text{for } N \in \mathbb{N}.$$

Then the following are true:

   (a) (Bessel's inequality)

$$\sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 \leq \|f\|_2^2.$$

   (b) (Best approximation)

$$\|f - S_N(f)\|_2 \leq \|f - g\|_2,$$

   for any $g \in \text{span}\{\phi_1, \dots, \phi_N\}$, for all $N \in \mathbb{N}$. Moreover, equality holds if and only if $g = S_N(f)$.

   (c) (Orthonormal basis)

$$\lim_{N\to\infty} \|f - S_N(f)\|_2 = 0$$

   if and only if the Parseval's identity holds:

$$\sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 = \|f\|_2^2.$$

   In either cases, $\{\phi_n\}_{n=1}^{\infty}$ is called an orthonormal basis in X.

3. Consider the space $\mathcal{R}[-\pi,\pi]$ with the "inner product"

$$\langle f,g\rangle := \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

Then we know that $\Phi := \{\frac{1}{\sqrt{2\pi}}e^{inx}\}_{n=-\infty}^{\infty}$ is an orthonormal basis. Thus the results in 2 hold, and the Parseval's identity takes the form

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

4. Similarly one can consider the space $\mathcal{R}[-1,1]$ with the "inner product"

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}dx.$$

Applying Gram-Schmidt process to the sequence of monomials $\{x^n\}_{n=0}^{\infty}$, we obtain an orthonormal set $\{\sqrt{\frac{2n+1}{2}}P_n\}_{n=0}^{\infty}$, where each P_n is a polynomial of degree n. The polynomials $\{P_n\}_{n=0}^{\infty}$ are called Legendre polynomials. They have a closed form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n[(x^2-1)^n]}{dx^n},$$

and satisfy the differential equations

$$[(1-x^2)P_n'(x)]' + n(n+1)P_n(x) = 0 \quad \text{for } n \geq 0.$$

Moreover one can show that $\{\sqrt{\frac{2n+1}{2}}P_n\}_{n=0}^{\infty}$ is an orthonormal basis, using Weierstrass Approximation Theorem. The Parseval's identity takes the form

$$\sum_{n=0}^{\infty} \frac{2n+1}{2}|\langle f,P_n\rangle|^2 = \int_{-1}^{1} |f(x)|^2 dx.$$

(This formula is written wrongly in the tutorial.)

Ex 1 Let f be a 2π-periodic function which is differentiable on $[-\pi,\pi]$ with f' integrable on $[-\pi,\pi]$. Show that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Ex 2 (Trigonometric series that is not a Fourier series.) Show that the trigonometric series

$$\sum_{n\geq 2} \frac{1}{\log n}\sin nx$$

converges for every x, yet it is not the Fourier series of a 2π-periodic function integrable on $[-\pi,\pi]$.