# Mathematical Analysis III

# Tutorial 10 (November 28)

The following were discussed in the tutorial this week:

1. We first recall Arzela-Ascoli Theorem:

   **Theorem** (Arzela-Ascoli). *Let $\mathcal{F}$ be a closed set in $C(K)$, where $K \subset \mathbb{R}^n$ is closed and bounded. Then $\mathcal{F}$ is compact $\Leftrightarrow$ $\mathcal{F}$ is bounded and equicontinuous.*

   **Remark.** *(a)* ***In the tutorial, I forgot to assume that $\mathcal{F}$ is closed in $C(K)$. Sorry.***

   *(b) The theorem can be easily generalized to $C(K)$, where $K$ is any compact metric space.*

   *(c) In showing the direction $\Leftarrow$, one only needs to assume that $\mathcal{F}$ is pointwise bounded, that is for all $x \in K$, there is $M_x > 0$ such that $|f(x)| \leq M_x$ for $f \in \mathcal{F}$.*

2. (HW 9, Q1) Let $K$ be a bounded convex subset of $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in $C(K)$ if there is a point $x_0 \in K$ and a constant $M > 0$ such that $|f(x_0)| \leq M$ for all $f \in \mathcal{F}$.

3. Let $\{a_n\}$ be a sequence of non-zero real numbers. Show that the sequence of functions
$$f_n(x) = \frac{1}{a_n}\sin(a_n x) + \cos(x + a_n)$$
has a subsequence that converges to a continuous function on every compact subset of $\mathbb{R}$.

4. **Definition:** A topological space $X$ is said to be a Baire space if it satisfies any one of the following equivalent properties:

   (a) Every countable union of closed subsets of $X$ with empty interior has empty interior.

   (b) Every countable union of dense open subsets of $X$ is dense in $X$.

   **Theorem** (Baire Category Theorem). *Every complete metric space is a Baire space.*

5. (a) Show that $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ are Baire spaces.

   (b) Show that $\mathbb{Q}$ is not a Baire space.

6. Show that $\mathbb{Q}$ cannot be expressed as a countable intersection of open subsets of $\mathbb{R}$.

7. Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that
$$\{x \in \mathbb{R} : f \text{ is continuous at } x\} = \mathbb{Q}.$$