THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3060 Mathematical Analysis III

Mid-term Examination

October 26, 2016

Answer all the questions.

Justify your steps and simplify your answers.

Notation: For functions f and g defined on the interval $[a, b]$, we denote $d_1(f, g) = \int_a^b |f(x) - g(x)|dx$ and $d_2(f, g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{1/2}$

(1) (25 points) Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ and show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots;$$
$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots.$$

(2) (25 points) Let a_n and b_n be the Fourier coefficients of some 2π-periodic Riemann integrable function f.

(a) Show that for all $r \in [0, 1)$,

$$a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

is uniformly convergent to some continuous function $f_r(x)$ on $[-\pi, \pi]$.

(b) Show that

$$f_r(x) = \int_{-\pi}^{\pi} \frac{1 - r^2}{2\pi(1 - 2r\cos\theta + r^2)} f(x + \theta) d\theta.$$

(P.T.O.)

(3) (25 points)

(a) Show that $d'(f,g) = d_2(f',g')$ is not a metric on $C^1_{2\pi}$ the set of continuously differentiable 2π-periodic functions.

(b) Show that $d'(f,g)$ is a metric on the subset $E = \{f \in C^1_{2\pi} : \int_{-\pi}^{\pi} f(x)dx = 0\}$.

(c) Show that $f_n \to f$ in (E, d') implies $f_n \to f$ in (E, d_2).

(d) Is the converse statement of (c) true? Justify your answer by a proof or a counterexample.

(4) (25 points)

(a) Let d and ρ be metrics on X. Suppose that ρ is stronger than d. Show that an open set in (X, d) is also open in (X, ρ).

(b) Is an open set in $(C[0,1], d_1)$ also open in $(C[0,1], d_2)$? Justify your answer by a proof or a counterexample.

(c) Is an open set in $(C[0,1], d_2)$ also open in $(C[0,1], d_1)$? Justify your answer by a proof or a counterexample.

(End)

(1) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \left[\frac{x^2}{2\pi}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{2}{\pi}\left\{ x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\,dx \right\}$$

$$= \frac{2}{\pi}\left\{ 0 + \left[\frac{\cos nx}{n^2}\right]_0^{\pi} \right\}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - 1\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \cancel{\frac{2}{\pi 2k}} = \begin{cases} 0 & \text{if } n = 2k \\ \dfrac{-4}{\pi(2k-1)^2} & \text{if } n = 2k-1 \end{cases} \quad k = 1, 2, 3.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx\,dx = 0. \quad (\text{even} \times \text{odd} = \text{odd})$$

$$\Rightarrow \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left\{ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos(2k-1)x \right\}$$

Lip. condition at $x=0$

$$\therefore \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

[illegible] $\rightarrow$ Parseval's [illegible]

$$\int_{-\pi}^{\pi} |x|^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$2\int_0^{\pi} x^2 dx = 2\pi \cdot \left(\frac{\pi}{2}\right)^2 + \pi \sum_{k=1}^{\infty} \frac{16}{\pi^2} \frac{1}{(2k-1)^4}$$

$$\left.\frac{2x^3}{3}\right|_0^{\pi} = \frac{\pi^3}{2} + \frac{16}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\left(\frac{2}{3} - \frac{1}{2}\right)\pi^3 = \frac{16}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\frac{1}{6} \pi^3 \cdot \frac{\pi}{16} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

(2) (a) By Riemann-Lebesgue $a_n, b_n \to 0$

$\Rightarrow$ a_n, b_n are bdd.

$\Rightarrow \exists M$ s.t. $|a_n| \leq M$

& $|b_n| \leq M$.

$\Rightarrow r^n |a_n \cos nx + b_n \sin nx| \leq 2M r^n \quad \forall x$

$\Rightarrow$ Since $0 \leq r < 1$ we have by M-test

$a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$

uniformly converges to some cts. function by M-test

(b) $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$

$S_N(f) =$

$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{N} r^n (\cos n\theta \cos nx + \sin n\theta \sin nx) f(\theta)$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} \{ \frac{1}{2} + \sum_{1}^{N} r^n \cos[n(\theta - x)] \} f(\theta) d\theta$

$\xi = \theta - x$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{N} r^n \cos n\xi\right\} f(x+\xi)\,d\xi$$

changing dummy index

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{N} r^n \cos n\theta\right] f(x+\theta)\,d\theta .$$

Note that $\quad 1 + 2\sum_{n=1}^{N} r^n \cos n\theta$

$$= 1 + 2\sum_{n=1}^{N} r^n \left(\frac{e^{in\theta} + e^{-in\theta}}{2}\right)$$

$$= 1 + \sum_{n=1}^{N}\left(r^n e^{in\theta} + r^n e^{-in\theta}\right)$$

$$\cancel{re^{i\theta}} \quad = 1 + \sum_{n=1}^{N}\left(re^{i\theta}\right)^n + \sum_{n=1}^{N}\left(re^{-i\theta}\right)^n$$

$$= \frac{1-(re^{i\theta})^{n+1}}{1-re^{i\theta}} + \frac{1-(re^{-i\theta})^{n+1}}{1-re^{-i\theta}} - 1$$

$$= \frac{(1-r^{n+1}e^{i(n+1)\theta})(1-re^{-i\theta}) + (1-r^{n+1}e^{-i(n+1)\theta})(1-re^{i\theta})}{1-re^{i\theta}-re^{-i\theta}+r^2}$$

$$= 1-r^{n+1}e^{i(n+1)\theta} - re^{-i\theta} + r^{n+2}e^{in\theta} + 1 - r^{n+1}e^{-i(n+1)\theta} - r$$

$$z+\cdots+z^N = z(1+z+\cdots+z^{N-1}) = \frac{z(1-z^N)}{1-z}$$

$$= 1+\frac{re^{i\theta}(1-r^Ne^{iN\theta})}{1-re^{i\theta}}+\frac{re^{-i\theta}(1-r^Ne^{-iN\theta})}{1-re^{-i\theta}}$$

$$= 1+\frac{re^{i\theta}}{1-re^{i\theta}}+\frac{re^{-i\theta}}{1-re^{-i\theta}} - r^{N+1}\left[\frac{e^{i(N+1)\theta}}{1-re^{i\theta}}+\frac{e^{-i(N+1)\theta}}{1-re^{-i\theta}}\right]$$

$$= \frac{(1-re^{i\theta})(1-re^{-i\theta})+re^{i\theta}(1-re^{-i\theta})+re^{-i\theta}(1-re^{i\theta})}{(1-re^{i\theta})(1-re^{-i\theta})}+O(r^{N+1})$$

$$= \frac{1-re^{i\theta}-re^{-i\theta}+r^2+re^{i\theta}-r^2+re^{-i\theta}-r^2}{1-re^{i\theta}-re^{-i\theta}+r^2}+O(r^{N+1})$$

$$= \frac{1-r^2}{1-2r\cos\theta+r^2}+O(r^{N+1})$$

$$\therefore S_N(f) = \int_{-\pi}^{\pi}\left[\frac{1-r^2}{2\pi(1-2r\cos\theta+r^2)}+O(r^{N+1})\right]f(x+\theta)\,d\theta$$

$$\cancel{= \int_{-\pi}^{\pi}\frac{1-r^2}{}}$$

integrable as $f\in\mathcal{R}$

$$\therefore \left|S_N(f)-\int_{-\pi}^{\pi}\frac{1-r^2}{2\pi(1-2r\cos\theta+r^2)}f(x+\theta)\,d\theta\right| \le O(r^{N+1})\to 0$$

as $N\to+\infty$

$$\therefore f_r(x) = \int_{-\pi}^{\pi}\frac{1-r^2}{2\pi(1-2r\cos\theta+r^2)}f(x+\theta)\,d\theta$$

(a) $d'(f,g)=0 \Rightarrow d_2(f',g')=0 \Rightarrow f'=g'$ since $f,g \in C^1_{2\pi}$

$\Rightarrow f = g + \text{constant}$

$\therefore$ d' is not a metric on $C^1_{2\pi}$

e.g. $d'(f, f+c) = d_2(f', f') = 0$, but $f \neq f+c$.

(b) $d'(f,g) = 0 \Rightarrow \cancel{f} \; f-g = \cancel{\text{const}} \; c$, const.

$f, g \in E \Rightarrow 0 = \int_{-\pi}^{\pi} f - \int_{-\pi}^{\pi} g = c$

$\therefore \quad f = g.$

$d'(f,g) \cancel{\geq 0} = d_2(f',g') \geq 0$

$d'(f,g) = \cancel{d'(g,f)} \; d_2(f',g') = d_2(g',f') = d'(g,f)$

$d'(f,g) = d_2(f',g') \leq d_2(f',h') + d_2(h',g')$

$= d'(f,h) + d'(h,g)$.

$\therefore$ d' is a metric on E.

(c) By Wirtinger's ineq. (since f' cts $\Rightarrow f^2 \,\&\, f'^2$ are integrable)

~~$\left(\int_{-\pi}^{\pi}(f-g)\right)^2 \leq \int_{-\pi}^{\pi}(f'-g')dx$~~

since $\int_{-\pi}^{\pi}(f-g) = 0$.

$$\int_{-\pi}^{\pi}(f-g)^2 dx \leq \int_{-\pi}^{\pi}(f'-g')^2 dx$$

$$d_2(f,g) \leq d'(f,g)$$

$\therefore$ $f_n \to f$ in (E, d') $\Rightarrow$ $d_2(f_n, f) \leq d'(f_n, f) \to 0$

$\therefore$ $f_n \to f$ in (E, d_2)

(d) No. Consider

$$f_n(x) = \frac{1}{n} \sin nx, \quad \forall n = 1, 2, 3 \cdots$$

Then ① $f \in C^1_{2\pi}$ & $\int_{-\pi}^{\pi} \frac{1}{n} \sin nx \, dx = 0 \therefore f_n \in E$.

② $$\int_{-\pi}^{\pi} f_n^2 = \frac{1}{n^2} \int_{-\pi}^{\pi} \sin^2 nx \le \frac{2\pi}{n^2}$$

$$\therefore \quad d_2(f_n, 0) \le \frac{\sqrt{2\pi}}{n} \to 0 \quad \text{as } n \to \infty$$

$$\therefore \quad f_n \to 0 \text{ in } (E, d_2).$$

However $$\int_{-\pi}^{\pi} f_n'^2 = \int_{-\pi}^{\pi} \cos^2 nx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 2nx) \, dx$$

$$= \pi$$

$$\therefore \quad d'(f_n, 0) = \left(\int_{-\pi}^{\pi} f_n'^2 \right)^{1/2} = \sqrt{\pi} \not\to 0$$

$$\therefore \quad f_n \not\to 0 \text{ in } (E, d').$$

4 (a) Let G be open in (X, d),

then $\forall x \in G$, $\exists \varepsilon > 0$ s.t. $B^d_\varepsilon(x) = \{y : d(y,x) < \varepsilon\} \subset G$

~~Since $d \le C\rho$ for some $C > 0$~~

Since ρ stronger than d, $\exists C > 0$ s.t.

$$d \le C\rho$$

$$\therefore \forall y \in B^\rho_{\frac{\varepsilon}{C}}(x) = \{y : \rho(y,x) < \tfrac{\varepsilon}{C}\},$$

$$d(y,x) \le C\rho(y,x) < C\frac{\varepsilon}{C} = \varepsilon$$

$$\therefore y \in B^d_\varepsilon(x)$$

$$\therefore B^\rho_{\frac{\varepsilon}{C}}(x) \subset B^d_\varepsilon(x) \subset G.$$

$\therefore$ G is open in (X, ρ).

(b) Yes! Since $\forall f, g \in C[0,1]$,

$$d_1(f,g) = \int_0^1 |f-g| \le \left(\int_0^1 |f-g|^2\right)^{1/2} \left(\int_0^1 1\right)^{1/2}$$

by Schwarz ineq.

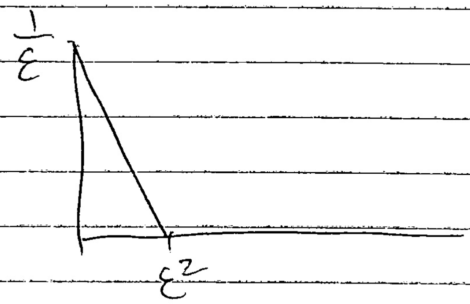
$$= d_2(f,g)$$

$\therefore$ d_2 is stronger than d_1,

By (a), open set in $(C[0,1], d_1)$ is open in $(C[0,1], d_2)$

(C) No:

Consider $f_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} - \frac{x}{\varepsilon^3} & \text{if } 0 \le x \le \varepsilon^2 \\ 0 & \text{if } \varepsilon^2 \le x \le 1 \end{cases}$



Then $f_\varepsilon \in C[0,1]$.

$$\left(\int_0^1 f_\varepsilon^2\right)^{1/2} = \left(\int_0^{\varepsilon^2} \left(\frac{1}{\varepsilon} - \frac{x}{\varepsilon^3}\right)^2 dx\right)^{1/2}$$

$$= \left(\varepsilon^3 \int_{-\frac{1}{\varepsilon}}^{0} y^2\, dy\right)^{1/2} \qquad \begin{cases} y = \frac{x}{\varepsilon^3} - \frac{1}{\varepsilon} \\ dy = \frac{dx}{\varepsilon^3} \end{cases}$$

$$= \left[\varepsilon^3 \left(\frac{y^3}{3}\right)_{-\frac{1}{\varepsilon}}^{0}\right]^{1/2}$$

$$= \left(\varepsilon^2 \frac{1}{3\varepsilon^3}\right)^{1/2} = \frac{1}{\sqrt{3}}$$

$$\int_0^1 |f_\varepsilon| = \int_0^{\varepsilon^2} \left(\frac{1}{\varepsilon} - \frac{x}{\varepsilon^3}\right) dx = \left.\frac{\varepsilon^2}{\varepsilon} - \frac{x^2}{2\varepsilon^3}\right|_0^{\varepsilon^2}$$

$$= \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Then $\cancel{f_\varepsilon \in}\ G := B^2_{\frac{1}{\sqrt{3}}}(0)$ is open in $(C[0,1], d_2)$.

By above $f_\varepsilon \in B^1_\varepsilon(0) \setminus B^2_{\frac{1}{\sqrt{3}}}(0)$, $\forall \varepsilon > 0 \Rightarrow B^1_\varepsilon(0) \not\subset B^2_{\frac{1}{\sqrt{3}}}(0)$

$\Rightarrow f \equiv 0 \in B^2_{\frac{1}{\sqrt{3}}}(0)$ is not an interior pt.

of G in $(C[0,1], d_1)$

$\therefore$ $G = B^2_{\frac{1}{\sqrt{3}}}(0)$ is <u>not</u> open in $(C[0,1], d_1)$