Solution 8

1. Let (X, d) be a metric space and $(Y, \|\cdot\|_Y)$ a Banach space. Show that the vector space $C_b(X;Y)$ consisting of all bounded, continuous mappings from X to Y forms a Banach space under the sup-norm
$$\|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y \ .$$
Remark: It reduces to $C_b(X)$ when $Y = \mathbb{R}$.

   Solution. It suffices to make some minor changes on the special case $C_b(X, \mathbb{R})$ described in the beginning of Chapter 3.

2. Let D be a dense set in the complete metric space X. Show that every uniformly continuous function defined on D can be extended to become a uniformly continuous function on X.

   Solution. Let f be uniformly continuous in D. For $\varepsilon > 0$, there is some δ such that $|f(u) - f(v)| < \varepsilon/2$ for $u, v \in D, d(u, v) < \delta$. For $x \in X$, we can pick $\{x_n\} \subset D, x_n \to x$. Then for large m, n, $d(x_m, x_n) < \delta$ so $|f(x_m) - f(x_n)| \leq \varepsilon/2$ and $\{f(x_n)\}$ is Cauchy. By the completeness of X we can define $F(x) = \lim_{n\to\infty} f(x_n)$. It is clear that this definition is independent of the sequence as long as it is in D and converges to x. Taking $x_n \equiv x$ when $x \in D$, we see that F coincides with f in D. For x, y now in X, we can find $\{x_n\}, \{y_n\} \subset D$ such that $x_n \to x, y_n \to y$. Let $d(x, y) < \delta$. Then $d(x_n, y_n) < \delta$ for large n, we get $|F(x) - F(y)| = \lim_{n\to\infty} |f(x_n) - f(y_n)| \leq \varepsilon/2$, done.

3. Let ℓ^∞ consist of all bounded sequences in $\mathbb{R}$. It is a Banach space under the norm $\|a\|_\infty = \sup_k |a_k|$ for $a = \{a_k\}$ in ℓ^∞. Show that this space is not separable.

   Solution. The proof is like the $B[a, b]$ case. We consider the family of bounded sequences which take 1 or 0 as their values. Then their distance is always equal to 1 (in supnorm). Let $\mathcal{B}$ be the collection of balls of radius 1/2 centered at each one of these sequences. They are mutually disjoint. As there are uncountably many of such balls and each dense set must intersect every one of them, no dense set could be countable. We conclude that ℓ^∞ is not separable.

4. Show that the closure of totally bounded set is bounded.

   Solution. Let S be a totally bounded set. Let $\varepsilon > 0$, there exists $\{x_i\}_{i=1}^n$ such that $\cup_{i=1}^n B(x_i, \frac{\varepsilon}{2})$ forms an open cover of S. Let $A = \cup_{i=1}^n B(x_i, \varepsilon)$. Let $y \in \bar{S}$, there exists $s \in S$ such that $d(s, y) < \varepsilon/2$ and since $s \in S$, there exists x_j s.t. $s \in B(x_j, \frac{\varepsilon}{2})$. Then, $d(y, x_j) \leq d(y, s) + d(s, x_j) < \varepsilon$ Hence, $\bar{S} \subset A$.

5. Show that every totally bounded set is bounded but the converse is not true.

   Solution. Let S be a totally bounded set. Let $\varepsilon > 0$, there exists $\{s_i\}_{i=1}^n$ such that $\cup_{i=1}^n B(s_i, \varepsilon)$ forms an open cover of S. Let x,$y \in S$. Then there exists s_j, s_k such that $x \in B(s_j, \varepsilon)$, $y \in B(s_k, \varepsilon)$. Then, $d(x, y) \leq d(x, s_j) + d(s_j, s_k) + d(s_k, y) \leq 2\varepsilon + \max\{d(s_u, s_v)|1 \leq u, v \leq n\} \leq (n + 2)\varepsilon$. i.e. S is bounded.

   Converse is not true. Consider discrete metric on $\mathbb{Z}$, $\mathbb{Z} \subset B(0, 2)$ and hence $\mathbb{Z}$ is bounded. But for all $n \in \mathbb{Z}$, $B(n, \frac{1}{2}) = \{n\}$ only. Hence $\mathbb{Z}$ cannot be covered by a finite collection of balls of radius $\frac{1}{2}$.