Solution 6

1. Show that an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $C[a,b]$ under L^2-producthas no convergent subsequence in the L^2-metric.

   Solution. Since $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal set, one has
   $$\int_a^b |\varphi_k - \varphi_j|^2\,dx = \int_a^b |\varphi_k|^2 + |\varphi_j|^2 - 2\varphi_k\varphi_j\,dx = 2,$$
   for any $k \neq j$. Hence it does not have any converging subsequence in L^2-distance.

   Some of you may use the fact that $f = \sum_{j=1}^{\infty} < f, \varphi_j >_2 \varphi_j$. No, we can't do this. It is true for the trigonometric series in L^2-sense but not true for all orthonormal sets.

2. Using open cover description, show that every continuous mapping of a compact set in some metric space (X,d) to another metric space (Y,ρ) is uniformly continuous.

   Solution. By continuity, for each x and $\varepsilon > 0$, there is some $\delta_x > 0$ such that
   $$\rho(f(y), f(x)) < \varepsilon/2, \quad \forall y \text{ with } d(y,x) < 2\delta_x.$$
   The ball $B_{\delta_x}(x)$ forms an open cover of X. By the compactness of X there is a subcover $B_{\delta_{x_k}}(x_k),\ k = 1, \cdots, N$. Let $\delta = \min\{\delta_{x_1}, \cdots, \delta_{x_N}\}$. For x, y satisfying $d(x,y) < \delta$, we first find $B_{\delta_{x_k}}(x_k)$ to contain x, then $d(x,x_k) < \delta_{x_k}$ and
   $$d(y,x_k) \leq d(y,x) + d(x,x_k) < \delta + \delta_{x_k} \leq 2\delta_{x_k}.$$
   It follows that
   $$\rho(f(y), f(x)) \leq \rho(f(y), f(x_k)) + \rho(f(x_k), f(x)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

3. Show that every continuous function from a compact metric space (X,d) to $\mathbb{R}$ attains its minimum and maximum.

   Solution. Let F be a continuous function from (X,d) to $\mathbb{R}$. By Problem 5, the image $F(X)$ is a compact set. Hence the number $m = \inf_{x\in X} F(x)$ is finite. Let $\{x_n\} \in X$ be a sequence such that $F(x_n) \to m$. Since X is compact, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in X$. One has $F(x) = \lim_{k\to\infty} F(x_{n_k}) = m$, due to the continuity of F. By the same arguments, F also attains its maximum.

4. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that $(C_b(X,d_\infty)$ forms a complete metric space,
   where $d_\infty(f,g)$=sup$|f-g|$.

   Solution. Let $\{f_n\}$ be a Cauchy sequence in $C_b(X)$. For $\varepsilon > 0$, there exists n_1 such that for any $m, n \geq n_1$, we have
   $$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon, \quad \forall x \in X. \tag{1}$$
   It shows that $\{f_n(x)\}$ is a numerical Cauchy sequence, so $\lim_{n\to\infty} f_n(x)$ exists. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We check it is continuous at x_0 as follows. By passing $m \to \infty$ in (1), we have
   $$|f(x)-f(x_0)| \leq |f(x)-f_{n_1}(x)|+|f_{n_1}(x)-f_{n_1}(x_0)|+|f_{n_1}(x_0)-f(x_0)| \leq 2\varepsilon+|f_{n_1}(x)-f_{n_1}(x_0)|.$$
   As f_{n_1} is continuous, there is some δ such that $|f_{n_1}(x) - f_{n_1}(x_0)| < \varepsilon$ for $x \in B_\delta(x_0)$. It follows that we $|f(x) - f(x_0)| < 3\varepsilon$ for $x \in B_\delta(x_0)$, so f is continuous at x_0. Now, letting $m \to \infty$ in (1), we get $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq n_1$, so $f_n \to f$ uniformly. In particular, it means f is bounded.

5. Let (X, d) be a metric space and $p \in X$ is a fixed point. Define for each $x \in X$, the function $f_x : X \to \mathbb{R}$ by
$$f_x(y) = d(y, x) - d(y, p).$$
   (a) Show that $f_x \in C_b(X)$, where $C_b(X)$ as in question 4.
   (b) Show that the mapping $\Phi : (X, d) \to (C_b(X), d_\infty)$ defined by $\Phi(x) = f_x \in C_b(X)$ is an isometric embedding.

   This approach is much shorter than the proof given in notes. However, it is not so inspiring.

   Solution.

   (a) From $|f_x(z)| = |d(z, x) - d(z, p)| \leq d(x, p)$, and from $|f_x(z) - f_x(z')| \leq |d(z, x) - d(z', x)| + |d(z', p) - d(z, p)| \leq 2d(z, z')$, it follows that each f_x is a bounded, uniformly continuous function in X.
   (b) $|f_x(z) - f_y(z)| = |d(z, x) - d(z, y)| \leq d(x, y)$, and equality holds taking $z = x$. Hence
$$\|f_x - f_y\|_\infty = d(x, y), \quad \forall x, y \in X.$$

6. Let T be a continuous self map on a complete metric space (X, d). Suppose that for some $k \geq 1$, T^k is a contraction. Show that T admits a unique fixed point.

   Solution. Since T^k is a contraction, there is a unique fixed point $x \in X$ such that $T^k x = x$. Then $T^{k+1}x = T^k T x = Tx$ shows that Tx is also a fixed point of T^k. From the uniqueness of fixed point we conclude $Tx = x$, that is, x is a fixed point for T. Uniqueness is clear since any fixed point of T is also a fixed point of T^k.

7. Consider maps from $\mathbb{R}$ to itself. Find an explicit example of a map satisfying $|f(x)-f(y)| < |x - y|$ but no fixed points.

   Solution. The function $f(x) = e^{-x} + x$ satisfies $|f(x) - f(y)| < |x - y|$ by mean value theorem. However it does not have any fixed point.