Pf: We may assume $E_j = \overline{E_j}$, $\forall j$ (by Note (ii) above.)

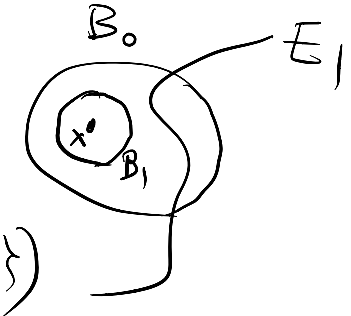
To show that $\bigcup_{j=1}^{\infty} E_j$ has empty interior, we only need to show that for any metric ball B_0, with radius r_0,

$$B_0 \cap \left(X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right) \right) \neq \phi .$$

As E_1 is nowhere dense, $\exists\, x \in B_0 \setminus E_1$.

Since E_1 is closed, there exists a closed ball $\overline{B}_1 = \{ y : d(y,x) \leq r_1 \}$ centered at x with radius

$\phi \neq$ $r_1 \leq \frac{r_0}{2}$ such that $\overline{B}_1 \cap E_1 = \phi$.



Next, as E_2 is nowhere dense & closed,

$\exists\, \overline{B}_2 \subset B_1$ $\left(B_1 = \{ y : d(y,x) < r_1 \} \right)$ ($\neq \phi$)

such that $\overline{B}_2 \cap E_2 = \phi$ and $r_2 \leq \frac{r_1}{2}$

(where r_2 = radius of $\overline{B}_2$)

and hence $\overline{B}_2 \cap E_1 = \phi$ as $\overline{B}_2 \subset B_1$ & $\overline{B}_1 \cap E_1 = \phi$.

Repeating this process,

we obtain a sequence of closed balls $\overline{B}_j \, (\neq \phi)$ such that

(a) $\overline{B}_{j+1} \subset B_j$,

(b) $r_{j+1} \leq \frac{r_j}{2} \leq \cdots \leq \frac{r_0}{2^{j+1}}$

(c) $\overline{B}_j \cap E_1 = \overline{B}_j \cap E_2 = \cdots = \overline{B}_j \cap E_j = \phi$.

For each $j \geq 1$, pick $x_j \in \overline{B}_j$.

By (a) & (b), we see that $\{x_j\}$ is a Cauchy sequence. Therefore, completeness of $(X, d) \Rightarrow$ $x_j \to x^*$ for some $x^* \in X$.

Clearly, $x^* \in \overline{B}_j \subset \cdots \subset B_0$. We claim that $x^* \in X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right)$:

Suppose not, then $x^* \in \bigcup_{j=1}^{\infty} E_j$

$$\Rightarrow \quad x^* \in E_j \text{ for some } j$$

$$\Rightarrow \quad x^* \in \overline{B}_j \cap E_j = \phi$$

which is a contradiction.

$\therefore \quad x^* \in X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right)$.

Hence $\quad x^* \in B_0 \cap \left(X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right) \right)$.

$\Rightarrow \quad B_0 \cap \left(X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right) \right) \neq \phi$

$\Rightarrow \quad \bigcup_{j=1}^{\infty} E_j$ has empty interior. ※

<u>Remarks</u>: (i) Baire Category Theorem $\Rightarrow$ Intersection of countably many <u>open dense</u> sets is again a dense set.

<u>Pf</u>: Let $\{G_j\}_{j=1}^{\infty}$ be <u>open dense</u> sets of X.

Then $\quad E_j = X \setminus G_j \quad$ is closed

& $\quad X \setminus E_j = G_j \quad$ is dense.

$\therefore \quad E_j$ is nowhere dense, $\forall j$.

$$\Rightarrow \quad X \setminus \left(\bigcup_{j=1}^{\infty} E_j \right) \text{ is dense}$$

$$\bigcap_{j=1}^{\infty} (X \setminus E_j) = \bigcap_{j=1}^{\infty} G_j$$

※

(ii) Note that the intersection of countably many open dense sets is dense, but not necessary open.

eg: Let $\mathbb{Q} = \{q_j\}_{j=1}^{\infty} \subset \mathbb{R}$ (rational numbers)

Then $D_k = \mathbb{R} \setminus \{q_1, \cdots, q_k\}$ is open dense sets in $\mathbb{R}$.

$\bigcap_{k=1}^{\infty} D_k = \mathbb{R} \setminus \mathbb{Q}$ is dense, but not open.

(iii) Baire Category Theorem $\Rightarrow$ it is impossible to decompose a complete metric space into a countable union of nowhere dense subsets (as $X \setminus (\bigcup_{j=1}^{\infty} E_j) \neq \emptyset$).

Def: A set in a metric space is called of first category if it can be expressed as a countable union of nowhere dense sets.

A set is of second category if its complement is of first category.

Notes: (i) By definition, any subset of a set of 1^{st} category is again of 1^{st} category.

(ii) Baire Category Theorem $\Rightarrow$ if X is complete, a set of 2^{nd} category is dense.

Prop 4.14 If a set in a <u>complete</u> metric space is of 1^{st} category, it cannot be of 2^{nd} category, and vice versa.

<u>Pf</u>: Let E be of 1^{st} category,

then $E = \bigcup_{k=1}^{\infty} E_k$ where E_k nowhere dense.

Suppose on the contrary that E is also of 2^{nd} category, then $X \setminus E$ is of 1^{st} category

$$\Rightarrow \quad X \setminus E = \bigcup_{k=1}^{\infty} F_k$$

for nowhere dense sets F_k, $k=1,2,\cdots\infty$.

Hence
$$\begin{aligned} X &= (X \setminus E) \cup E \\ &= \left(\bigcup_{k=1}^{\infty} F_k\right) \cup \left(\bigcup_{k=1}^{\infty} E_k\right) \\ &= \bigcup_{k=1}^{\infty} (F_k \cup E_k) \end{aligned}$$

as E_k, F_k are nowhere dense $\Rightarrow$ $E_k \cup F_k$ is nowhere dense (Ex)

$\therefore$ This contradicts the Baire Category Thm. ※

<u>eg</u>: $\mathbb{R}$ complete, $\mathbb{Q} = \bigcup \{q_i\}$ is of 1^{st} category.

& $\mathbb{R} \setminus \mathbb{Q}$ is of 2^{nd} category.